UNDERSTANDING THE QUANTUM BEHAVIOUR OF MATTER AS A DERIVED PROPERTY. II. RECOVERY OF THE QUANTUM FORMALISM

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ABSTRACT

The atomic problem is approached a new within the framework of SED, allowing the zeropoint field to be modified by its interaction with matter. By requiring the complete matter-field system to satisfy a principle of stability against fluctuations, that characterizes the stationary regime, one is led in a natural form to the Heisenberg equations of motion for the description of the mechanical part.

I. BASIS FOR A NEW SED

This is the second part of our contribution to this Conference, and should be read as the continuation of part I, which deal with a relationship between Planck's constant and cosmological constants involving the large-numbers relation $N_2^2 \sim N_3$. As anticipated in the introduction contained in part I, we now present a modified version of SED that is suited to deal with the atomic problem and leads to results which are fully consistent with the quantum formalism.

To make a long story short, we start as usual with the equation of motion for a particle subject to the simultaneous action of an external binding force and the zeropoint radiation field. In an approximate form, which is sufficient for the present analysis, one may write the corresponding Abraham-Lorentz equation (usually renamed in the context of SED after Braffort and Marshall, two pioneers of the theory)

$$m\ddot{x} = m\tau\, \dddot{x} + F(x) + eE(x,t). \qquad (1)$$

where $\mathbf{E}(x,t)$ represents the electric zeropoint field and is of course a stochastic variable with zero mean; the magnetic force term is neglected, and the constant factor that appears in the radiation reaction term is the time parameter $\tau = 2e^2/3mc^3$. Given the external force $\mathbf{F}(x)$, the problem is simply stated: look for the stationary solutions of equation (1) and find out if they have something to do with the corresponding quantum description, after assigning adequate statistical properties to the (vacuum) field $\mathbf{E}(x,t)$.

This kind of program has been carried out for some particular problems with results that go from satisfactory to excellent. For example, excellent results are obtained in the study of van der Waals and Casimir forces, the diamagnetic properties of electrons, etc.; we speak only of satisfactory results for the harmonic oscillator, because the excited levels, even if they are there, appear in an entirely formal way, more or less as they do in usual quantum theory; thus, the physics remains obscure. As was said before, when dealing however with the hydrogenic atom, the results of such a direct approach to the problem are disastrous: no bound ground state is predicted.

A more careful consideration of the above examples shows that the different problems have not been treated with equity. Indeed, one usually takes for granted the statistical properties of the vacuum field, and uses these same properties from the beginning to the end of the calculation, as if the vacuum field were exactly *the same* for all problems, and equal to the free field. However, in other cases, for instance when dealing with macroscopic bodies, as in the calculation of Casimir forces, one takes into consideration the effects of the boundaries (which are of course made of atoms), so that the vacuum field is adjusted to the

specific problem. Another such example is the calculation of the modification of the background field within a dielectric material, in equilibrium with the atoms of the material. One may argue that a similar consideration should be applied to *all* cases, including the single atom. Indeed, the complete problem refers to the coupled matter-field system, and in principle both parts of the system become mutually affected by the interaction , as soon as they start to interact. Of course, to state the problem in these terms practically amounts to surrendering, because it becomes intractable; however we may be somewhat less ambitious and consider the effects of the adjustment of the field to the *final* equilibrium situation, by looking for a self-consistent stationary solution of equation (1) that satisfies appropriate conditions, assuming that the matter-field system reaches a situation of dynamical equilibrium –which is supposed to be precisely the situation dealt with by the quantum formalism. In other words, one would expect a close correspondence between the self-consistent stationary solutions of SED and the stationary solutions of quantum mechanics.

As a first approach in this direction, we consider a particular kind of stationary solutions which are characterized by the fact that they are the less random possible solutions, as independent of the *specific* realization of the random field, or, what amounts to the same, as stable against small perturbations, as possible.[1,2] These stable stationary solutions turn out to be characterized by two properties, namely, they correspond precisely to the quantum mechanical solutions given by the Heisenberg equations of motion, and moreover some 'relevant' field components of the modified vacuum field end up having correlated phases, which indicates a central aspect in which this field differs from the free vacuum field, characterized by statistically independent phases.

II. SELF-CONSISTENT SED SOLUTION FOR A SIMPLE NONLINEAR PROBLEM

According to the previous discussion and contrary to standard practice in usual SED, along the treatment of the mechanical part of the SED problem we consider the statistical properties of the vacuum field initially unknown, and let them become determined by the requirements of the system itself under stationary conditions, that is, once the quantum regime has been attained. For arbitrarily small time intervals the conditions of the system are largely arbitrary, and may even be inconsistent with the usual quantum behavior; this is the reason we consider a system that has already evolved towards the quantum regime. We represent the field as a time Fourier transform in the form

$$E(x,t) = \sum_{\sigma,k} \tilde{E}_k\sigma(x) a^0_{k\sigma} e^{-i\omega_k t} + \text{c.c.}, \quad k = \omega_k / c. \tag{2}$$

The functions $\tilde{E}_k\sigma(x)$ carry all the space dependence of the field, whereas the coefficients $a_k\sigma$ are the stochastic variables that determine its statistical properties. Strictly speaking, the sum must be understood as an integration, because the field contains all frequencies from zero to infinity; the point is that, as we shall soon see, there are some specific frequencies to which the system responds with particular intensity (we call them the relevant frequencies); the rest of components taken together play the role of a noise, with little influence on the determination of the basic equilibrium properties. This noise is already being neglected in (2) but it must be reintroduced when performing a more detailed analysis (it gives rise, for instance, to radiative corrections[1,2]). In what follows we further simplify the treatment by avoiding everywhere the question of the polarization.

In the long-wavelength approximation one neglects the spatial dependence of the field; the $\tilde{E}_k\sigma$ are therefore taken as constants, whose value is determined by the energy of the field (see equation (5) below). Since the Fourier components $\tilde{E}_k a^0_k$ correspond to an infinite number of modes, each with the same average energy $\frac{1}{2}\hbar\omega_k$, the energy per mode of each of these components has an extremely sharp distribution, as follows from the central limit theorem, and the random amplitudes $a^0_{kj} = \left(a^0_k\right)_j$ have surely a (very nearly) Gaussian distribution sharply peaked around ω_k; thus, for all practical purposes one can consider them as having fixed amplitudes but random phases. For the *free vacuum field* they are taken as statistically independent, so that

$$\left\langle a_{ki}^{0}\right\rangle = 0, \quad \left\langle a_{ki}^{0} a_{k'i'}^{0*}\right\rangle = \delta_{kk'}\delta_{ii'}, \tag{3}$$

$$a_{ki}^{0} = e^{i\varphi_{ki}}, \tag{4}$$

with the random phases φ_{ki} uniformly distributed over $(0,2\pi)$. Equation (3) is alright only for the noise (which we are not taking into account), but not for the relevant frequencies, the properties of which are to be generated by the theory itself (the general rule will be given in section III below). Equation (4), in contrast, holds in general; all the stochasticity of the field is now expressed in the random phases. With the selection (4), we must take

$$\tilde{E}_k = i\sqrt{\frac{\pi\hbar\omega_k}{V}} \tag{5}$$

for the energy per mode bo be $\frac{1}{2}\hbar\omega_k$. In the above expressions a superindex '0' has been added to the field amplitudes, with the purpose of introducing the following notation (we omit from now on the Cartesian indices wherever they are unneccessary):

$$a_k \equiv a_k(t) = a_k^0 e^{-i\omega_k t}. \tag{6}$$

Note that $a_k(t)$ and a_k^0 have the same statistical properties.

The Braffort-Marshall equation that will be used for the analysis of the quantum regime reads therefore, for a one-dimensional problem,

$$m\ddot{x} = m\tau\dddot{x} + F(x) + e\sum_k \tilde{E}(\omega) a\omega^0 e^{-i\omega t} + c.c., \tag{7}$$

with $\tilde{E}_k$ given by (5) and random amplitudes of the form (4).

It should be clear that since a partial averaging is being performed over all modes of a given frequency, the description afforded by equation (7) refers not to an individual particle, but to an ensemble of equivalent particles, each one acted on by a specific mode of such frequency. Alternatively, we may refer to a partly averaged behaviour of a given particle. Thus, the ensuing theory will be *essentially statistical*, just as is the case with quantum mechanics. Also, the emerging mechanical variables are to be seen as partly averaged random variables, and not as strictly individual local variables; in particular, the correlations between variables may be poorly described.

To be above limitations and features of the ensuing theory, we must add that the detailed behaviour in terms of the single field modes has not only becomes hidden, but is irretrievably lost for the description to follow (hence also for the quantum description). We cannot, any more recover a fully deterministic picture by 'adding the hidden variables' or any similar simple procedure; it would not be a matter of embedding the ensuing description into a larger theory, but of constructing a new one from scratch.

To solve equation (7) in general[1,2] one starts by writing both the stationary x(t) and the force as Fourier series in the form

$$x = \sum_k \tilde{z}_k(\omega) e^{-i\omega_k t} + cc, \tag{8}$$

$$F(x) = \sum_k \tilde{\Phi}_k(\omega) e^{-i\omega_k t} + cc, \tag{9}$$

so that from (7) one gets the set of equations

$$m(-\omega_k^2 + i\tau\omega_k^3)\tilde{z}_k = \tilde{\Phi}_k + e\tilde{E}_k a_k^0. \tag{10}$$

Both $\tilde{z}(\omega)$ and $\tilde{\Phi}(\omega)$ may depend on arbitrary combinations of the random amplitudes a_k^0 of the different relevant frequencies ω_k, which are stochastic variables; further, $\tilde{\Phi}(\omega)$ depends in general on $\tilde{z}(\omega)$ in a complicated way, with the obvious exception of the linear-force problem. Observe that no speficific relation among the relevant frequencies is being considered, as is done beforehand, for example, in the treatment of classical multiply periodic systems.

To simplify the exposition it is convenient to consider a simple nonlinear example that contains already the typical features of the general case. A natural example is the anharmonic oscillator consisting of a linear oscillator plus a cubic force term and governed by the equation

$$\ddot{x} + \omega_0^2 x + \kappa x^3 = \tau \dddot{x} + \frac{e}{m} E(t). \tag{11}$$

The core of the problem resides in the nonlinear part of the force; its Fourier coefficient is, from equations (8) and (9),

$$\tilde{\Phi}_k^{(3)} = \lim_{T\to\infty} \frac{\kappa}{2T} \int_{-T}^{T} x^3(t) e^{i\omega_k t} dt = \kappa \sum_{n'n''n'''} \tilde{z}_{n'} \tilde{z}_{n''} \tilde{z}_{n'''}, \tag{12}$$

where the sum extends over the set of indices for which the frequencies satisfy the condition

$$\omega_k = \omega_{n'} + \omega_{n''} + \omega_{n'''} \tag{13}$$

coming from the delta function that is generated by the time integration of $e^{i(\omega_{n'}+\omega_{n''}+\omega_{n'''})t}$ extended to infinite limits. Note that this condition reduces the number of independent Fourier indices to two for a given frequency ω_k.

With the dissipative effect of the radiation reaction term $m\tau \dddot{x}$ fully taken into account, only one stationary solution of the form (8) should exist, corresponding to the ground state if the appropriate stationary field is used. However, in the radiationless approximation –which corresponds to the quantum mechanical description–, the system admits more than one stationary solution, and usually an infinite number of them. An extra index (α) is therefore required to distinguish between different solutions:

$$x\alpha(t) = \sum_n \tilde{z}_{\alpha n} e^{-i\omega_{n\alpha} t} + c.c. \tag{14}$$

Note the reverse order of the subindices of ω in the exponent; we have adopted this convention to adjust the final results to the normal conventions of quantum mechanics. It will be seen below that the two indices (here taken as different and denoted by n and α) play actually a symmetric role; therefore Greek letters will be used for both. Further, it will turn out that $\omega_{\beta\alpha} = -\omega_{\alpha\beta}$ (see equation (55)), so that one may change the sign of the time exponents by inverting the order of the indices, and write equation (14) in the form

$$x\alpha(t) = \sum_\beta \tilde{z}_{\alpha\beta} e^{i\omega_{\alpha\beta} t} + c.c. \tag{15}$$

The Fourier coefficient of the nonlinear term of frequency $\omega_{\alpha\beta}$ is then

$$\tilde{\Phi}^{(3)}(\omega)\Big|_{\omega_{\alpha\beta}} = \kappa \sum (\tilde{z}_{\mu'\upsilon'}\tilde{z}_{\mu''\upsilon''}\tilde{z}_{\mu'''\upsilon'''})_{\omega_{\alpha\beta}}, \tag{16}$$

where the summation is to be performed over the set of indices such that

$$\omega_{\alpha\beta} = \omega_{\mu'\nu'} + \omega_{\mu''\nu''} + \omega_{\mu'''\nu'''}, \tag{17}$$

according to (13). Although the number of subindices has increased, it should be kept in mind that in equation (16) there is only a double sum, which means that the additional indices must be related to the summation indices, as will be clarified below.

With (16), equation (10) becomes for this problem

$$(-\omega_{\alpha\beta}^2 + i\tau\omega_{\alpha\beta}^3 + \omega_0^2)\tilde{z}_{\alpha\beta} + \kappa \sum (\tilde{z}\tilde{z}''\tilde{z}''')_{\alpha\beta} = \frac{e}{m}\tilde{E}(\omega_{\alpha\beta})a_{\alpha\beta}^0. \tag{18}$$

Now given the structure of the right-hand-side term in this equation, it is convenient to rewrite the Fourier amplitudes of the position coordinate and the force in the form

$$\tilde{z}_{\alpha\beta} = \tilde{x}_{\alpha\beta}a_{\alpha\beta}^0. \tag{19}$$

$$\tilde{\Phi}_{\alpha\beta} = \tilde{F}_{\alpha\beta}a_{\alpha\beta}^0. \tag{20}$$

In the particular case of the linear oscillator ($\kappa = 0$), the resulting coefficients $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ are independent of the $a_{\alpha\beta}^0$, and are thus nonrandom numbers; but with $\kappa \neq 0$ the situation is different; in this case it follows from equation (18) that the coefficient $\tilde{x}_{\alpha\beta}$ depends on the set amplitudes $\{a_{\alpha\beta}^0$ in a quite complicated way, for on substituting (19) there appear products of three a^0's containing all those frequencies which combine to give just $\omega_{\alpha\beta}$, as demanded by equation (17). Explicitly, and on dividing by the common factor $a_{\alpha\beta}^0$, one gets the set of coupled stochastic equations

$$(-\omega_{\alpha\beta}^2 + i\tau\omega_{\alpha\beta}^3 + \omega_0^2)\tilde{x}_{\alpha\beta} + \kappa \sum (\tilde{z}\tilde{z}''\tilde{z}''')_{\alpha\beta} = \frac{(a^{0'}a^{0''}a^{0'''})\alpha\beta}{a_{\alpha\beta}^0} = \frac{e}{m}\tilde{E}(\omega_{\alpha\beta}). \tag{21}$$

These equations determine the *response amplitudes* $\tilde{x}_{\alpha\beta}$ and *characteristic frequencies* $\omega_{\alpha\beta}$ for the problem; these latter correspond to what we have termed relevant frequencies.

The solutions of (21) are functions of the random amplitudes $a_{\alpha\beta}^0$, and thus are in principle stochastic numbers by themselves; they represent a different set of stationary solutions for every realization of the field. Here is where we deviate from the original SED approach. We observe that for *certain* random fields there are solutions to (21) that are nonrandom numbers and thus independent of the specific realization of the field. These particularly simple solutions occur when the set of equations that determine them contains no random coefficients, i.e., when all explicit dependence on the a^0's vanishes from the equations, which in the case under study occurs only when the following equality holds for *each* relevant frequency,

$$(a^{0'}a^{0''}a^{0'''})_{\alpha\beta} = a_{\alpha\beta}^0, \tag{22}$$

except for a possible constant factor of proportionality (which must be set equal to 1, as was already done here). For those fields that satisfy these conditions, the characteristic (Fourier) frequencies $\omega_{\alpha\beta}$ and the

corresponding response amplitudes $\tilde{x}_{\alpha\beta}$ of the stationary states of motion are essentially insensitive to the fluctuation of the random variables a^0, and hence independent of the specific realization. They are thus remarkably stable solutions. We propose to consider seriously these solutions, determined by demanding the frequencies $\omega_{\alpha\beta}$ to become *non-stochastic.*

Let us now investigate the consequences of this demand. Firstly, with (22), the system of algebraic equation (21) reduces to

$$(-\omega_{\alpha\beta}^2 + i\tau\omega_{\alpha\beta}^3 + \omega_0^3)\tilde{x}_{\alpha\beta} + \kappa\sum(\tilde{x}\tilde{x}''\tilde{x}''')_{\alpha\beta} = \frac{e}{m}\tilde{E}(\omega_{\alpha\beta}), \tag{23}$$

whose solutions $(\tilde{x}_{\alpha\beta}, \omega_{\alpha\beta})$ are obviously deterministic. A further important consequence is obtained as follows.

Introducing (19) into (15) we write with the help of equations (17), (20) and (22), in a synthetic notation,

$$x^3 = \sum \tilde{z}'\tilde{z}''\tilde{z}''' e^{i(\omega'+\omega''+\omega''')t} = \sum \left(\sum \tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta} \left(a^{0'}a^{0''}a^{0'''}\right)_{\alpha\beta} e^{i(\omega'+\omega''+\omega''')t}$$

$$= \sum \left(\sum \tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta} a^0_{\alpha\beta} e^{i\omega_{\alpha\beta}t} = \sum (\tilde{x}^3)_{\alpha\beta} e^{i\omega_{\alpha\beta}t}, \tag{24}$$

where the element $(\tilde{x}^3)_{\alpha\beta}$ involves a double summation over indices such that (17) and (22) are satisfied; in the previous shortand notation this reads

$$(\tilde{x}^3)_{\alpha\beta} = \left(\sum \tilde{x}'\tilde{x}''\tilde{x}'''\right)_{\alpha\beta}. \tag{25}$$

The detailed meaning of this and other similar rules involving constrained sums will be discussed shortly. What should be noted here is that a force which is nonlinear in x has become a linear function of the field amplitudes $a^0_{\alpha\beta}$, as shown by equation (24). This is a most important result: despite the presence of nonlinearities, the system *responds linearly to the field* and behaves as a set of linear oscillators of frequency $\omega_{\alpha\beta}$ and amplitude $\tilde{x}_{\alpha\beta}$. We stress that no linear approximation is being made, but it is the system's response to the field that is linear under the assumed stability conditions, with sharply defined frequencies and response coefficients determined by the nonlinear equations (23). This is the reason why we call the present theory *linear* SED.

To see the meaning and implications of the above results let us use them to recast equation (23) in the algebraic form

$$-m\omega_{\alpha\beta}^2\tilde{x}_{\alpha\beta} = \tilde{F}_{\alpha\beta} - im\tau\omega_{\alpha\beta}^3\tilde{x}_{\alpha\beta} + e\tilde{E}(\omega_{\alpha\beta}), \tag{26}$$

with $\tilde{F}_{\alpha\beta} = (\tilde{F}(\tilde{x}))_{\alpha\beta}$ including of course the nonlinear terms. For the specific case of the anharmonic oscillator the force coefficients are given by

$$\tilde{F}_{\alpha\beta} = -\omega_0^2\tilde{x}_{\alpha\beta} - \kappa\sum(\tilde{x}\tilde{x}''\tilde{x}''')_{\alpha\beta}, \tag{27}$$

where the (double) sum in the triple product of $\tilde{x}$ must be such that equation (17) is satisfied.

Observe from (15) and (19) that one can use alternatively time-dependent coefficients defined by

$$\tilde{x}_{\alpha\beta}(t) = \tilde{x}_{\alpha\beta} e^{i\omega_{\alpha\beta} t}, \tag{28}$$

$$\tilde{F}_{\alpha\beta}(t) = \tilde{F}_{\alpha\beta} \exp(i\omega_{\alpha\beta} t), \tag{29}$$

$$\tilde{E}_{\alpha\beta}(t) = \tilde{E}_{\alpha\beta} \exp(i\omega_{\alpha\beta} t), \tag{30}$$

so that

$$\frac{d^2}{dt^2} \tilde{x}_{\alpha\beta}(t) = -\omega_{\alpha\beta}^2 \tilde{x}_{\alpha\beta}(t) \tag{31}$$

and so on, and equation (26) once multiplied by $\exp(i\omega_{\alpha\beta} t)$ becomes

$$m \frac{d^2 \tilde{x}_{\alpha\beta}(t)}{dt^2} = \tilde{F}_{\alpha\beta}(t) + m\tau \frac{d^3 \tilde{x}_{\alpha\beta}(t)}{dt^3} + e\tilde{E}_{\alpha\beta}(t). \tag{32}$$

This is a set of (nonlinear) deterministic equations of motion for the $\tilde{x}_{\alpha\beta}(t)$; all random quantities have vanished from the description, once the fundamental effects of the random radiation field have been taken into account. We can thus take now the radiationless approximation by writing the above equation to zero order in *e*,

$$m \frac{d^2 \tilde{x}_{\alpha\beta}(t)}{dt^2} = \tilde{F}_{\alpha\beta}(t). \tag{33}$$

Despite its form this (unperturbed) equation is *not* a classical equation of motion, owing to the specific meaning and algebraic properties of the terms $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$. In fact, it is a Heisenberg equation of motion (as will be shown), with solutions that correspond to those of the quantum mechanical description of stationary states. It is astonishing that equation (33), which can be (and is) taken as the fundamental law for the problem, does not contain any element whatsoever reminding us of its stochastic origin, and no trace whatsoever of the background field that generates and sustains the stationary solutions.

III. THE GENERAL BOUND PROBLEM

We now turn our attention to the solution of the general equation

$$m\ddot{x} = m\tau \dddot{x} + F(x) + e\sum_k \tilde{E}_k a_k^0 e^{-iw_k t} \tag{34}$$

in the quantum regime, and show that the results of the previous section are applicable in general. For simplicity, in equation (34) the c.c. terms of the zero-point field have been embodied in the sum, which now runs over positive and negative frecuencies. Following equations (15) and (19) one writes.

$$x\alpha(t) = \sum \beta \tilde{z}_{\alpha\beta} e^{i\omega_{\alpha\beta} t} = \sum \beta \tilde{x}_{\alpha\beta} a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta} t} \tag{35}$$

and introduces this expansion into equation (34), to obtain

$$x\alpha = -\frac{e}{m} \sum \beta \frac{\tilde{E}_{\alpha\beta}}{\Delta(\omega_{\alpha\beta})} = a_{\alpha\beta}^0 e^{i\omega_{\alpha\beta} t}, \tag{36}$$

where

$$\Delta(\omega) = \omega^2 + \frac{1}{m}\frac{\tilde{\Phi}(\omega)}{\tilde{z}(\omega)} - i\tau\omega^3 \tag{37}$$

still depends on the random amplitudes. As before, $\tilde{\Phi}(\omega)$ represents the Fourier transform of the external force. We assume that all relevant singularities are simple isolated poles, and that they give the dominant contributions to $x\alpha(t)$. The system is assumed to behave resonantly at the frequencies corresponding to the poles; the smallness of the parameter τ entering in the radiation reaction force guarantees the sharpness of the resonance, so that the contributions from the poles are clearly dominant. For bounded motions, the equations for the poles $\Delta(\omega) = 0$, or

$$\frac{1}{m}\frac{\tilde{\Phi}(\omega)}{\tilde{z}(\omega)} = -\omega^2 + i\tau\omega^3, \tag{38}$$

is satisfied only for certain (discrete, and in general stochastic) frequencies. For instance, for the harmonic oscillator of natural frequency ω_0 there are poles at + ω_0 and - ω_0, but in the case of particles bound by nonlinear forces, the values of ω at the poles will in general depend on the state of motion. Further, since for all frequencies ω of interest $|\tau\omega| \ll 1$, we take the radiationless approximation, so that equation (38) becomes

$$-\frac{1}{m}\frac{\tilde{\Phi}_{\alpha\beta}(\omega)}{\tilde{z}_{\alpha\beta}(\omega)} = \omega_{\alpha\beta}^2. \tag{39}$$

Now we introduce the requirement of nonrandom values for the characteristic frequencies $\omega_{\alpha\beta}$. The ratio $\tilde{\Phi}_{\alpha\beta}/\tilde{z}_{\alpha\beta}$ must then be independent of $a^0_{\alpha\beta}$, and equations (35), (36) and (39) lead to $\tilde{z}_{\alpha\beta}$ and $\tilde{\Phi}_{\alpha\beta}$ linear in $a^0_{\alpha\beta}$, so that one can write

$$\tilde{z}_{\alpha\beta} = \tilde{x}_{\alpha\beta}a^0_{\alpha\beta}, \quad \tilde{\Phi}_{\alpha\beta} = \tilde{F}_{\alpha\beta}a^0_{\alpha\beta} \tag{40}$$

with $\tilde{x}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta}$ nonrandom coefficients, related by the system of algebraic equations contained in (39), namely,

$$\tilde{F}_{\alpha\beta} = -m\omega_{\alpha\beta}^2\tilde{x}_{\alpha\beta}. \tag{41}$$

>From this expression and the second one in (40) it follows that the Fourier transform of the external force is a linear function of the stochastic amplitudes; the linear response to the field is thus extended to any binding external force. It is remarkable that this general property follows as a consequence of the demand of nonrandom values for the characteristic frequencies $\omega_{\alpha\beta}$ of the stationary solutions.

Let us now investigate the properties that the field amplitudes $a^0_{\alpha\beta}$ must have for equations (40) to nold. We assume that the external force can be expressed as a power series in x; this leads to a sum of terms containing any number n of factors of $\tilde{z}_{\mu\nu}$, of the type

$$\tilde{z}_{\lambda_1\mu_1}\tilde{z}_{\lambda_2\mu_2}\cdots\tilde{z}_{\lambda_n\mu_n} = \tilde{x}_{\lambda_1\mu_1}\tilde{x}_{\lambda_2\mu_2}\cdots\tilde{x}_{\lambda_n\mu_n}a^0_{\lambda_1\mu_1}a^0_{\lambda_2\mu_2}\cdots a^0_{\lambda_n\mu_n}, \tag{42}$$

each of which should correspond to a fixed frequency, say $\omega_{\alpha\beta}$, so that

$$\left(\tilde{z}^n\right)_{\alpha\beta} = \left(\tilde{x}^n\right)_{\alpha\beta}a^0_{\alpha\beta}. \tag{43}$$

It follows the *any* product of *relevant* amplitudes a^0 must reduce to a single a^0, so that the product terms remain linear in a^0,

$$a^0_{\lambda_1\mu_1} a^0_{\lambda_2\mu_2} \cdots a^0_{\lambda_n\mu_n} = a^0_{\alpha\beta}. \tag{44}$$

Of the 2n indices appearing on the left hand side, two are fixed (α and β) and n – 1 are summation indices (due to the implicit δ-functions, as in equation (12)), so that n – 1 indices remain free; but since this product of a^0's should be just the required $\alpha^0_{\alpha\beta}$, and not another independent random amplitude, the indices must repeat themselves (otherwise independent random phases would appear). In particular, for n = 2 one can write either

$$a^0_{\alpha\beta} a^0_{\mu\mu} = a^0_{\alpha\beta} \tag{45}$$

or

$$a^0_{\alpha\beta} a^0_{\mu\beta} = a^0_{\alpha\beta} \tag{46}$$

for arbitrary α, β and μ. From the first of these equations we get

$$a^0_{\mu\mu} = 1, \tag{47}$$

whereas the iteration of (46) to arbitrary n gives:

$$a^0_{\alpha\mu_1} a^0_{\mu_1\mu_2} \cdots a^0_{\mu_{n-1}\beta} = a^0_{\alpha\beta}. \tag{48}$$

With the a^0's having constant magnitude equal to 1,

$$a^0_{\alpha\beta} = e^{i\varphi_{\alpha\beta}}, \tag{49}$$

equation (48) means that the phases must satisfy

$$\varphi_{\alpha\beta} = \varphi_{\alpha\mu_1} + \varphi_{\mu_1\mu_2} + \cdots + \varphi_{\mu_{n-1}\beta}, \text{ modulo } 2\pi \tag{50}$$

and, as follows from equations (46) and (45),

$$\varphi_{\mu\mu} = 0, \qquad \varphi_{\alpha\beta} = -\varphi_{\beta\alpha}, \tag{51}$$

It terms of the amplitudes, the latter constraint implies the relations

$$a^0_{\alpha\beta} = \left(a^0_{\beta\alpha}\right)^* = \left(a^0_{\beta\alpha}\right)^{-1}, \tag{52}$$

As follows from equation (17), the frequencies of the time coefficients corresponding to the a^0's will in their turn satisfy the equality

$$\omega_{\alpha\beta} = \omega_{\alpha\mu_1} + \omega_{\mu_1\mu_2} + \cdots + \omega_{\mu_{n-1}\beta} \tag{53}$$

and, in particular,

$$\omega_{\alpha\alpha} = \omega_{\alpha\beta} + \omega_{\beta\alpha}. \tag{54}$$

Hence with $(\alpha\alpha)$ denoting the field mode of zero frequency (a mode which is actually absent from the pure radiation field), $\omega_{\alpha\alpha} = 0$, the above equation implies the important symmetry property:

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{55}$$

which was used in advance in connection with equation (15).

The general solution to equation (50), taking into account (51), is

$$\varphi_{\alpha\beta} = \varphi\alpha - \varphi\beta \mod 2\pi, \tag{56}$$

with $\varphi\alpha$ and $\varphi\beta$ independent random phases uniformly distributed over the interval $(0,2\pi)$, so that $\varphi_{\alpha\beta} = \varphi\alpha - \varphi\beta \mod 2\pi$, is also uniformly distributed over the same interval. Thus the amplitudes have the general form

$$a^0_{\alpha\beta} = e^{i(\varphi\alpha - \varphi\beta)}, \tag{57}$$

which shows that the statistically independent random quantities are not the amplitudes $a^0_{\alpha\beta}$ with the combined index $(\alpha\beta)$, but the single-index phases $\varphi\alpha$. The characteristic frequencies share this important property of separability, i.e.,

$$\omega_{\alpha\beta} = \Omega\alpha - \Omega\beta, \tag{58}$$

where $\Omega\alpha$ are (nonrandom) numbers. One thus gets for the Fourier coefficient of a typical term $(\tilde{x}^n)$ corresponding to frequency $\omega_{\alpha\beta}$:

$$(\tilde{x}^n)_{\alpha\beta} a^0_{\alpha\beta} = \sum_{\mu_i} \tilde{z}_{\alpha\mu_1} \tilde{z}_{\mu_1\mu_2} \cdots \tilde{z}_{\mu_{n-1}\beta} = \left(\sum_{\mu_i} \tilde{x}_{\alpha\mu_1} \tilde{x}_{\mu_1\mu_2} \cdots \tilde{x}_{\mu_{n-1}\beta} \right) a^0_{\alpha\beta} \tag{59}$$

where the sum is performed over all allowed values of the (repeated) intermediate indices. One recognized here the multiplication rule for matrices:

$$(\tilde{x}^n)_{\alpha\beta} = \sum_{\mu_i} \tilde{x}_{\alpha\mu_1} \tilde{x}_{\mu_1\mu_2} \cdots \tilde{x}_{\mu_{n-1}\beta}, \tag{60}$$

so that the solution arrived at is naturally expressed in terms of matrices. For example, $\tilde{x}_{\lambda\mu}$ is the $\lambda\mu$-element of a matrix $\hat{x}$, and so on.

IV. THE HEISENBERG EQUATIONS OF MOTION

The result just obtained can be used to write the equations of motion in matrix form. For this purpose, recall that one may associate the time factor $\exp(i\omega_{\alpha\beta}t)$ of $\tilde{z}_{\alpha\beta}$ either with $\tilde{x}_{\alpha\beta}$, as in (28), or with $a_{\alpha\beta}$ (see (6)):

$$\tilde{x}_{\alpha\beta} a^0_{\alpha\beta} e^{i\omega_{\alpha\beta}t} = \tilde{x}_{\alpha\beta}(t) a^0_{\alpha\beta} = \tilde{x}_{\alpha\beta} a_{\alpha\beta}(t),$$

where

$$\tilde{x}_{\alpha\beta}(t) = \tilde{x}_{\alpha\beta} e^{i\omega_{\alpha\beta}t}, \; a_{\alpha\beta}(t) = a^0_{\alpha\beta} e^{i\omega_{\alpha\beta}t} \tag{61}$$

and use whatever expression is convenient in each case. In particular, from equations (33) and (60) one has in the radiationless approximation

$$m\frac{d^2\hat{x}(t)}{dt^2} = \hat{F}(\hat{x}), \tag{62}$$

where $\hat{x}(t)$ and $\hat{F}(\hat{x})$ are now time-dependent matrices with elements $\tilde{x}_{\alpha\beta}(t)$ and $\tilde{F}_{\alpha\beta}(\hat{x})$, respectively. To complete the description we define a matrix $\hat{p}(t)$ with elements that follow from the time derivative of equation (28),

$$\tilde{p}_{\alpha\beta}(t) = im\omega_{\alpha\beta}\tilde{x}_{\alpha\beta}(t), \tag{63}$$

or in matrix notation,

$$\hat{p} = m\frac{d\hat{x}}{dt}. \tag{64}$$

This definition is suitable for making contact with quantum mechanics, which is a zero-order theory as regards the radiative terms and hence unable to distinguish between the mechanical moment and the canonical moment with respect to the zero-point field (recall equations (32) and (33)). Combining with equation (62) it follows then that

$$\frac{d\hat{p}}{dt} = \hat{F}(\hat{x}). \tag{65}$$

Equations (64) and (65) are evidently the Heisenberg equations of motion, and $\hat{x}_{\alpha\beta}(t)$ are the elementary oscillators of matrix mechanics. The matrix algebra of quantum mechanics follows therefore as *the algebra that guarantees stable, nonrandom values for the characteristic frequencies of the stationary* SED *system in the radiationless and long-wavelength approximation.*

V. THE SCALE OF QUANTUM PHENOMENA

Observe that the above equations of motion (64), (65) do not yet fully determine $\hat{x}_{\alpha\beta}$ and $\omega_{\alpha\beta}$. In writing the stationary solutions (39) in the form of (35), only the positions of the poles were taken into account, without ever really solving the equation (34) which contains the full information, and, in particular, fixes x(t) in terms of Planck's constant. This means that we have to come back to the equations that describe the *complete* SED system before the radiationless approximation is made and the zero-point field is dropped altogether, in order to fix the scale of the solutions.

Instead of dealing with the complicated original equation of motion, however, a simpler procedure can be used, observing that what is lacking in the above formalism to complete the full system of Heisenberg equations is the value of some fundamental commutator, such as, e.g., $[\hat{x},\hat{p}]$. This is not the place to enter into details, which requiere a lenghty consideration, so we refer the reader to the cited references and content ourselves here with a sketch of the procedure and the final result. The basic idea is to consider the translation into the above language, of the fundamental Poisson brackets of the original theory. It is possible to demonstrate that in particular, the SED Poisson bracket $[x_i,p_j] = \delta_{ij}$ transforms in the new language into

$$-\sum\beta\omega_{\alpha\beta}\left|\tilde{x}_{\alpha\beta}\right|^2 = \frac{\hbar}{2m}. \tag{66}$$

This result can be identified as the Thomas-Reiche-Kuhn sum roule of quantum mechanics, which is just the quantization rule $[\hat{x},\hat{p}]=i\hbar$ expressed in terms of matrix elements. It is through this (and similar) results that $\hbar$ enters into the scheme.

Now we take advantage of the separability of $\omega_{\alpha\beta}$ expressed in equation (58), to write

$$\dot{\tilde{x}}_{\alpha\beta} = i\omega_{\alpha\beta}\tilde{x}_{\alpha\beta} = i(\Omega\alpha - \Omega\beta)\tilde{x}_{\alpha\beta} = -i\sum\mu(\tilde{x}_{\alpha\mu}\Omega\beta\delta_{\mu\beta} - \Omega\alpha\delta_{\alpha\mu}\tilde{x}_{\mu\beta}) \tag{67}$$

which can be recast as follows, with H the Hamiltonian of the mechanical system,

$$i\hbar\,\dot{\tilde{x}}_{\alpha\beta} = \sum\mu(\tilde{x}_{\alpha\mu}\tilde{H}_{\mu\beta} - \tilde{H}_{\alpha\mu}\tilde{x}_{\alpha\beta}), \tag{68}$$

whence a comparison gives

$$\tilde{H}_{\alpha\beta} = \hbar\Omega\alpha\delta_{\alpha\beta}. \tag{69}$$

This result shows that the matrix representing H in the present formalism is diagonal, which means that $H\alpha$ is not random,

$$H\alpha = \sum\mu H_{\alpha\mu}a_{\alpha\mu} = \hbar\Omega\alpha a_{\alpha\alpha} = \hbar\Omega\alpha, \tag{70}$$

and further, that the $\Omega\alpha$ introduced via equation (58) is proportional to $H\alpha$, identified in quantum mechanics as the energy of the particle in state α,

$$\varepsilon\alpha \equiv \langle H\alpha\rangle = H\alpha = \hbar\Omega\alpha. \tag{71}$$

>From this it follows that equation (58) is Bohr's formula for the transition frequencies,

$$\hbar\omega_{\alpha\beta} = \varepsilon\alpha - \varepsilon\beta. \tag{72}$$

Hence the characteristic or relevant frequencies of SED coincide with the transition frequencies of quantum mechanics. Analogously, from the above relations it follows that the response amplitudes $\tilde{x}_{\alpha\beta}$ are the transition amplitudes of quantum mechanics.

VI. HILBERT-SPACE FORMALISM

A correspondence has been established between the description of linear SED and quantum mechanics, via the Heisenberg equations. Now it is a relatively easy matter to further develop the new description, until a direct contact with the usual Hilbert-space formalism is reached. As a practical means to achieve this we introduce the *a-representation*, as follows.

Consider a set of square matrices $\hat{a}^{\alpha\beta}$, each of which has only the element $\alpha\beta$ different from zero,

$$\left(\hat{a}^{\alpha\beta}\right)_{\mu\nu} = a_{\alpha\beta}\delta_{\alpha\mu}\delta_{\beta\nu}. \tag{73}$$

The coefficients $a_{\alpha\beta} \equiv a_{\alpha\beta}(t)$ are given by (57) and (61); thus,

$$a_{\alpha\beta} = e^{i\varphi_{\alpha\beta}+i\omega_{\alpha\beta}t} = e^{i(\varphi\alpha-\varphi\beta)+i(\Omega\alpha-\Omega\beta)t} = e^{i(\varphi\alpha-\Omega\alpha)t}e^{-i(\varphi\beta+\Omega\beta)t}. \tag{74}$$

Note that the off-diagonal elements have random phases, whereas for $\alpha = \beta$ the phase is zero. A product of two of such matrices gives, as follows from (73) and (74),

$$(\hat{a}^{\alpha\beta}\hat{a}^{\gamma\delta})_{\mu\nu} = a_{\alpha\delta}\delta_{\alpha\mu}\delta_{\beta\gamma}\delta_{\delta\nu} = \delta_{\beta\gamma}(\hat{a}^{\alpha\delta})_{\mu\nu}. \tag{75}$$

The fact that this product differs from zero only for $\beta = \gamma$ makes these matrices especially suited for the present purposes, for they can be used as a basis to write the matrix representing an arbitrary dynamical variable. For instance for the variable x we write

$$\hat{x} = \sum_{\alpha,\lambda} \tilde{x}_{\alpha\lambda}\hat{a}^{\alpha\lambda} = \sum \alpha\hat{x}\alpha, \tag{76}$$

where

$$\hat{x}\alpha = \sum \lambda \hat{x}_{\alpha\lambda}\hat{a}^{\alpha\lambda}; \tag{77}$$

the matrix elements of $\hat{x}$ are then just $\hat{x}_{\alpha\lambda} = \hat{x}_{\alpha\lambda}a_{\alpha\lambda}$. Then from (75) and (76) one gets for instance for the square of $\hat{x}$

$$\left(\hat{x}^2\right)_{\mu\nu} = \sum \lambda \tilde{x}_{\mu\lambda}\tilde{x}_{\lambda\nu}a_{\mu\lambda}a_{\lambda\nu} = \left(\tilde{x}^2\right)_{\mu\nu}a_{\mu\nu}. \tag{78}$$

The matrix x^2 is thus again a linear combination of the $\hat{a}$'s, with coefficients $\left(\hat{x}^2\right)_{\mu\nu} = \sum \lambda \tilde{x}_{\mu\lambda}\tilde{x}_{\lambda\nu}$. Consequently the operator $\hat{x}$ reproduces the matrix properties which the variable x must possess according to the discussion above. The same applies of course to any other observable, which means that it applies to any variable in the quantum regime that can be expressed in the linear form (35).

Now observe from equation (74) that the matrix $\hat{a}^{\alpha\beta}$ can be written as the product of two vectors, namely, a column vector $|\alpha\rangle$ of the form

$$|\alpha\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a\alpha \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix} a\alpha \tag{79}$$

and a row vector $\langle\beta|$ which is the adjoint of $|\beta\rangle$, $\langle\beta| = \left(|\beta\rangle\right)^{+}$, and is given by

$$\langle\beta| = (0 \quad 0 \quad \cdots \quad a\beta^* \quad \cdots) = (0 \quad 0 \quad \cdots \quad 1 \quad \cdots)a\beta^*, \tag{80}$$

with

$$a\alpha = e^{i(\varphi\alpha + \Omega\alpha t)}. \tag{81}$$

These vectors have as a many components as there are different indices α, which normally means a (denumerable) infinity of them. The only element of $|\alpha\rangle$ which is different from zero is in row α, i.e., $(|\alpha\rangle)\lambda = a\alpha\delta_{\alpha\lambda}$, and therefore

$$a\alpha a\beta^* = a_{\alpha\beta}, \tag{82}$$

is agreement with equation (74). Finally, from equations (73) and (79) follows the factorization rule

$$\hat{\alpha}^{\alpha\beta} = |\alpha\rangle\langle\beta|. \tag{83}$$

The vectors $|\alpha\rangle$ form a complete orthonormal basis, as follows from the equations

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta},\ \Sigma\,\alpha|\alpha\rangle\langle\alpha| = \Sigma\,\alpha\hat{a}^{\alpha\alpha} = \hat{1}. \tag{84}$$

They thus span the Hilbert space of the states of the system, and an observable f can be represented by any one of the following expressions:

$$\hat{f} = \sum \alpha\hat{f}\alpha = \sum_{\alpha,\beta} \tilde{f}_{\alpha\beta}\hat{a}^{\alpha\beta} = \sum_{\alpha,\beta} \tilde{f}_{\alpha\beta}|\alpha\rangle\langle\beta|, \tag{85}$$

with

$$\hat{f}_{\alpha\beta} = \langle\alpha|\hat{f}|\beta\rangle. \tag{86}$$

In particular for the Hamiltonian we have, as follows from (70),

$$\hat{H} = \sum \alpha\hat{H}_{\alpha\alpha}\hat{a}^{\alpha\alpha} = \sum \alpha\varepsilon\alpha|\alpha\rangle\langle\alpha|. \tag{87}$$

It is interesting to note that the vectors $|\alpha\rangle$ do not involve the $\omega_{\alpha\beta}$, but the quantities $\varepsilon\alpha/\hbar = \Omega\alpha$; the transition to a Hilbert-space formulation in terms of bras and kets has had the effect of shifting the accent from the relevant frequencies to the energy eigenvalues for the stationary states, and from the field amplitudes $a_{\alpha\beta}$ to the vector elements $a\alpha$. The proposed mathematical transformation has thus changed the conceptual framework into an entirely different one, in which the main objects are vectors in a Hilbert space and eigenvalue equations. Note further that in expressions such as (79) and (80) the factors $a\lambda$ contain random phases that remain hidden in the usual Hilbert-space formulation.

In the a-representation the dynamical equation take the form (see equations (61-66))

$$m\frac{d\hat{x}}{dt} = \hat{p}, \qquad \frac{d\hat{p}}{dt} = \hat{F} \tag{88}$$

and

$$[\hat{x},\hat{p}] = i\hbar\hat{1}. \tag{89}$$

Once more the time dependence can be attributed to $\tilde{x}$ by the writing $\sum\tilde{x}_{\alpha\beta}(t)\hat{a}^{\alpha\beta}$, or else to $\tilde{a}$ by writing $\sum\tilde{x}_{\alpha\beta}\hat{a}^{\alpha\beta}(t)$ and using equation (61). The passage from the first expression to the latter is equivalent to a transition from the Heisenberg to the Schrödinger picture.

VII. CLOSING COMMENTS

At this point it is appropriate to comment briefly on some of the features and implications of the theory just sketched.

The characteristics of the self-consistent solution show that the coupling between the atomic processes and the field due to radiation, generates in the long run phase correlations between the components of the

neighboring field, and between this field and the atomic motions, as was anticipated by Theimer and Peterson.[3] The method followed, though obviously successful, has the important shortcoming that the self-consistent solution must be accepted as a matter of fact. It is quite clear that without the use of an auxiliary principle or hypothesis it would have been impossible to identify the solution, due to the high complexity of the mathematical problem. The central question of deriving the solution from first principles remains open.

As already stated, an exact and detailed solution of the full problem would *not* lead to the present description; this is only reached after performing a series of approximations and simplifications., In the transition to such an approximate theory, some attributes of the starting description, as that of being genuinely statistical and local realistic, become much weakened and adopt their quantum guise. This can be identified as the main reason for the difficult interpretative issues characteristic of the quantum description. It is here in particular where we find an explanation to questions such as why the quantum formalism gives an incomplete and seemingly noncausal account of the behavior of mechanical systems: quantum mechanics appears from the present point of view not as a fundamental theory of matter, but as a derived, approximate, asymptotic theory. And approximate physical theories may not satisfy the same rigorous requirements that fundamental theories are supposed to fulfil; this is particularly true in regard to consistency with first principles.

Even if still unfinished, the theory presented allows already for a certain reinterpretation of some quantum issues. Of special interest is the finding that quantum operators such as $\hat{x}$, $\hat{p}$ and so on, should be interpreted as referring not to a single particle, but to the subensemble constructed through the coarse-graining process, and then in a highly abstract form, distant from any direct empirical meaning. The present theory is as intrinsically nonrelativistic as it is intrinsically statistical, and quantum mechanics inherits these peculiarities. In particular, the extended belief that the quantum variables can be readily identified with those describing the individuals, and with a meaning directly suggested by their classical counterpart, is not supported by this theory. Also, the fact that the theory is not constructed around the notion of trajectory, does not mean that individual trajectories do not exist, nor that the possibility of constructing a space-time description is cancelled forever. Simply, neither the present formulation nor quantum mechanics are such a theory.

Although the basic equations for the linear SED system in the quantum regime are stochastic by nature, they have been recast in nonrandom terms, which happen to be just those of matrix mechanics. In such a cryptic description (in terms of the Heisenberg equations of motion) the elements responsible for the stochasticity –the field amplitudes $a_{\alpha\beta}$- have vanished completely, resulting in a seemingly fully deterministic picture. This simple observation explains by itself much of the enduring interpretative problems of the usual quantum mechanics.

A question that invites us to indulge in further speculation refers to the possibility that under certain circunstances the system responds with random frequencies (as would be the case if the demand of detailed balance were removed), in which case the situation would be more chaotic than the one represented by usual quantum states. It is clear that for this to happen the system must leave the quantum regime, but it is unclear whether such a process means merely a return to a classical (stochastic) behaviour, or whether some new behaviour arises.

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