

# THE PHASES OF TWO-DIMENSIONAL QED AND QCD

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## ABSTRACT

The semi-classical phase structure of two-dimensional QED and QCD are briefly reviewed. The non-abelian theory is reformulated to closely resemble the Schwinger model. It is shown that, contrary to the abelian theory, the phase structure of two-dimensional QCD is unaffected by the structure of the theta vacuum. We make parallel calculations in the two theories and conclude that massless Schwinger model is in the screening and the massive theory is in the confining phase, whereas both massless and massive QCD are in the screening phase.

## 1. INTRODUCTION

Massless Schwinger model is an exactly solvable theory [1]. The phase structure of this theory has been studied extensively. It is well-established that the theory is in the Higgs or screening phase. On the other hand, massive Schwinger model is not an exactly solvable theory. Nevertheless, it has also been studied intensely and it is known that, under certain approximations, the theory is confining. There are various ways of establishing the phases of the Schwinger model. A rather simple method, which are shall demonstrate here, is to use the bosonised version of the theory and introduce external probe charges into the system. For a semi-classical theory, the interchange potential binding the test particles can be easily computed. Classically, the Coulomb potential is expected to rise linearly with the inter-charge separation. However, in the bosonised theory, vacuum polarization effect can shield the probe charges. As a result, in the massless Schwinger model, the confining Coulomb interaction is replaced by a screening potential. The massive theory, on the other hand, survives the polarization effects and is in the confining phase.

The same method can be applied to two-dimensional QCD [1]. Although an exact bosonisation formulae is not available for the non-abelian theory and the available bosonisation methods are perturbative in the mass parameter [9], the techniques developed for Schwinger model can be used to infer informations about the phase of two-dimensional QCD. We introduce external probe colour charges into the theory and evaluate their inter-charge potential. Unlike two-dimensional QED, both the massless and massive non-abelian theories are in the Higgs phase.

## 2. SCHWINGER MODEL

we start with the lagrangian for two-dimensional QED [4],

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \psi(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi. \quad (1)$$

This lagrangian can be re-written in terms of the bosonic variables, by using Mandelstam bosonisation formula [8]

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad (2)$$

$$\psi_1 \sim : e^{i\beta\phi + \frac{i}{\beta}\bar{\phi}} : \quad (3)$$

$$\psi_2 \sim e^{-i\beta\phi + \frac{i}{\beta}\tilde{\phi}} \quad (4)$$

The bosonised lagrangian is,<sup>1</sup>

$$L = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + e^{\mu\nu}A_\mu\partial_\nu\phi + m^2\gamma(\cos(2\sqrt{\pi}\phi) - 1), \quad (5)$$

where  $\gamma$  is a normalisation constant [7]. Next, we place external probe charges  $q$  and  $-q$  at  $L/2$  and  $-L/2$ . For the purpose of evaluating the inter-charge potential, it suffices to restrict ourselves to static fields (i.e.  $\partial_0 = 0$ ). The static lagrangian, incorporating the probe charges, is

$$L = \frac{1}{2}(\partial_1 A_0)^2 - \frac{1}{2}(\partial_1 \phi)^2 + m^2\gamma[\cos(2\sqrt{\pi}\phi) - 1] + \frac{e}{\sqrt{\pi}}A_0\partial_1\phi + A_0q[\delta(x - L/2) - \delta(x + L/2)] \quad (6)$$

The equations of the motion corresponding to the above lagrangian are

$$\frac{e}{\sqrt{\pi}}\partial_1\phi + q(\delta(x - L/2) - \delta(x + L/2)) - \partial_1^2 A_0 = 0 \quad (7)$$

$$-2\sqrt{\pi}m^2\gamma\sin(2\sqrt{\pi}\phi) - \frac{e}{\sqrt{\pi}}\partial_1 A_0 + \partial_1^2 \phi = 0 \quad (8)$$

The equation of motion of  $A_0$  can be integrated to give an expression for the scalar field in terms of the electric field  $E$  ( $\partial_1 A_0$ ), i.e.,

$$\phi = \frac{\sqrt{\pi}}{e}[\partial_1 A_0 - q(T(x - L/2) - T(x + L/2)) - \alpha], \quad (9)$$

where  $T$  is the step function and  $\alpha$  is the integration constant<sup>2</sup>. This can be inserted into the equation of motion for  $\phi$  to yield,

$$\partial_1^2 \tilde{E} - \frac{e^2}{\pi}\tilde{E} - \frac{eq}{\sqrt{\pi}}(T - \bar{T}) - 2\sqrt{\pi}m^2\gamma\sin\left(2\sqrt{\pi}\tilde{E} - \frac{2\pi\alpha}{e}\right) = 0 \quad (10)$$

where

$$\tilde{E} = \frac{\sqrt{\pi}}{e}[E - q(T - \bar{T})], \quad (11)$$

$\theta = 2\pi\alpha/e$  is the theta vacuum and  $(T - \bar{T})$  denotes  $(T(x - L/2) - T(x + L/2))$ .

In order to obtain the potential binding  $q$  and  $-q$ , we solve the above equation to first find the inter-charge electric field. The equation can be solved for two different cases: when the dynamical fermions are massless and when they are massive.

<sup>1</sup> Unlike the original lagrangian (1), the bosonised lagrangian does not describe a purely classical system. In the bosonisation procedure, one takes into account the vacuum polarization effects by evaluating the contributions from the one-loop vacuum functionals.

<sup>2</sup> Note that in four dimensions, this integration constant is zero. However, in two dimensions energetics allow for a non-vanishing background electric field [6].

## 2.1. Massless Schwinger model

For massless dynamical fermions, the equation of motion for  $\tilde{E}$  (10) reduces to

$$\partial^2 \tilde{E} - \frac{e^2}{\pi} \tilde{E} - \frac{e}{\sqrt{\pi}} q(T - \bar{T}) = 0. \quad (12)$$

This equation is exactly solvable and its solutions are

$$\tilde{E}_I = a \exp\left(\frac{-e}{\sqrt{\pi}} x\right), \quad x > \frac{L}{2} \quad (13)$$

$$\tilde{E}_{II} = b \exp\left(\frac{e}{\sqrt{\pi}} x\right), \quad x < \frac{L}{2} \quad (14)$$

$$\tilde{E}_{III} = c \exp\left(\frac{-e}{\sqrt{\pi}} x\right) + d \exp\left(\frac{e}{\sqrt{\pi}} x\right) + \frac{\sqrt{\pi} q}{e}, \quad -\frac{L}{2} < x < \frac{L}{2} \quad (15)$$

The solutions and their derivatives can be matched at the boundaries to easily obtain the unknown coefficients. Having obtained  $\tilde{E}$ , we use the expression (11) to obtain the electric field. The inter-charge potential  $V(L)$  can then be simply obtained by integrating the electric field over the inter-charge separation<sup>3</sup>, i.e.,

$$V(L) = -q \int_{-L/2}^{L/2} E_{III} dx = \frac{q^2 \sqrt{\pi}}{2e} \left(1 - e^{-eL/\sqrt{\pi}}\right) \quad (16)$$

The expression for the inter-charge potential shows that for small inter-charge separations, the potential rises linearly with  $L$ , as is expected from a classical coulomb potential. However, as the inter-charge separation is increased the polarization effects set up<sup>4</sup> and lead to the screening of the probe charges. For very large inter-charge separations, the potential eventually reaches the constant value  $q^2 \sqrt{\pi}/2e$ .

So far, the charge of the probe particles has not been specified. Therefore, even the screening of the fractional probe charges by integer dynamical charges is allowed. This is puzzling since one would expect the confinement to prevail in such cases. The situation can be clarified by evaluating the conserved charge  $Q$  associated with the integer dynamical charges. This charge is given in terms of the conserved current  $\partial_1 \phi$ , i.e.,

$$Q = \int_{x_1}^{x_2} dx \partial_1 \phi. \quad (17)$$

To unravel the mechanism of screening, we consider the screening of one of the probe charges and evaluate the conserved charge along the axis from 0 to  $\infty$  (i.e., a solitonic configuration.) That is,

$$Q = \phi(\infty) - \phi(0) = -q. \quad (18)$$

<sup>3</sup> This can be shown to be equivalent to evaluating the change in the hamiltonian caused by the probe charges (see [5], Chapter 10).

<sup>4</sup> Recall that we have taken into account the polarization effects in the bosonisation scheme. In replacing the original fermionic lagrangian by the bosonic lagrangian, one incorporates the contributions from the one-loop Feynman diagrams in the bosonic action.

This shows that although the dynamical charges are of integer values, the charge associated with their solitonic configuration can be non-integer. Specifically, this charge is opposite to the charge of the probe particles and accounts for the shielding phenomena.

## 2.2. Massive Schwinger model

For massive dynamical fermions, the Schwinger model is not exactly solvable. The equation of motion (10) is non-linear and can only be solved after expanding the sine-terms<sup>5</sup>. In this approximation, the equation of motion (10) reduces to

$$\partial_1 \tilde{E} - \left( \frac{e^2}{\pi} 4\pi m^2 \gamma \right) \tilde{E} - \left( \frac{e}{\sqrt{\pi}} q(T - T) + 2\sqrt{\pi} m^2 \gamma \theta \right) = 0 \quad (19)$$

This equation resembles (12) for the massless Schwinger model and can be solved in the same manner. By using the expression (11) relating  $\tilde{E}$  to the electric field, we obtain the electric field and subsequently the inter-charge potential. This is now given by

$$V(L) = \frac{e^4}{2\pi(e^2/\pi + 4\pi m^2 \gamma)^{3/2}} \left( 1 - e^{-\sqrt{(e^2/\pi + 4\pi m^2 \gamma)}L} \right) + \frac{e}{2} \left( 1 - \frac{e^2}{e^2 + 4\pi^2 m^2 \gamma} \right) \left( q - \frac{e\theta}{\pi} \right) L \quad (20)$$

Therefore, for the massive fermions, the inter-charge potential has both a screening and a confining term. However, for long separations the confinement term dominates. In addition, for integer probe charges and for  $\theta = \pi$  a phase transition occurs; the confinement term disappears and the screening phase is restored. This can be explained by recalling that the theta vacuum, which was introduced in equation (9) as an integration constant, is basically a non-vanishing background electric field. As the theta angle is increased, pair production sets up helping the screening of the probe charges. This continues until the net electric field falls below the threshold required for pair production. This circle repeats itself and therefore the dynamics of the system is a periodic function of the theta angle.

## 2.3. Two-dimensional QCD

In the preceding sections, we have studied the screening and confining phases in two-dimensional QED. We saw that although massless Schwinger model is in the screening phase, the massive theory exhibit confinement. In this section, we ask the same question for two dimensional QCD and examine whether the conclusions drawn for two-dimensional QED can be generalized to its non-abelian counterpart.

The action of two-dimensional QCD is

$$S = \int d^2x \left[ -\frac{1}{4} + \text{tr} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_i^f (i \not{\partial} \delta^{ij} - e A^{ij}) \psi_j^f - m \delta^{ij} \bar{\psi}_i^f \psi_j^f \right], \quad (21)$$

where  $i, j$  are the usual colour indices and  $f = 1, \dots, k$  is a flavour quantum number. Unlike the Schwinger model, for which an exact bosonisation scheme (3) exist, two-dimensional QCD cannot be exactly bosonised. However a perturbative bosonisation scheme by means of which the massless theory is first bosonised and the mass term is introduced and bosonised perturbatively exists [9]. Writing down the generating function for

<sup>5</sup> The validity of this approximation can be checked by evaluating the argument of the sine, using the approximate solution for the electric field. One verifies that, for a large mass to charge ratio, the argument of the sine is much smaller than  $\pi/4$ . Thus, the expansion of the sine term is only meaningful for  $m \gg e$ .

the above lagrangian and integrating out the fermions in the path integral measure leads to the well-known Wess-Zumino-Witten action

$$S_{\text{eff}} = \sum_f \Gamma[g_f] - (c_v + k)\Gamma[\Sigma] + k\Gamma[\beta] + S_{\text{YM}} + m^2 \int d^2z \text{tr} \left[ \sum_f (g_f \Sigma^{-1} \beta + g_f^{-1} \Sigma \beta^{-1}) \right], \quad (22)$$

where

$$S_{\text{YM}} = \int d^2z \left[ \frac{1}{2} (\partial_+ A_0)^2 + \lambda A_0 (\beta^{-1} i \partial_+ \beta) \right], \quad (23)$$

$$\Gamma[g] = \int \frac{d^2x}{8\pi} \text{tr} (\partial_\mu g^{-1} \partial^\mu g) + \int \frac{d^3y}{12\pi} \epsilon^{\alpha\beta\gamma} \text{tr} (g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g), \quad (24)$$

the light-cone coordinate is  $x^+$ ,  $\lambda = \frac{e}{2\pi} (c_v + k)$  and  $c_v$  is given by  $c_v = f^{ad} = f_{abc} f^{abd}$  which vanishes for the abelian group<sup>6</sup>. The  $g$  field is the gauge-invariant bosonic field corresponding to the original fermionic excitations. The field  $\Sigma$  is a negative-metric field and the fields  $\beta$  and  $A_0$  are the massive sector fields. The equations of motion corresponding to this action are

$$\frac{1}{4\pi} \partial_+ (g_f \partial_- g_f^{-1}) = m^2 (g_f \Sigma^{-1} \beta - \beta^{-1} \Sigma g_f^{-1}), \quad f = 1 \dots k, \quad (25)$$

$$-\frac{(c_v + k)}{4\pi} \partial_+ (\Sigma \partial_- \Sigma^{-1}) = m^2 \sum_{f=1}^k (\Sigma g_f^{-1} \beta^{-1} - \beta g_f \Sigma^{-1}), \quad (26)$$

$$-\frac{k}{4\pi} \partial_- (\beta^{-1} \partial_+ \beta) + i\lambda [\beta^{-1} \partial_+ \beta, A_0] + i\lambda \partial_+ A_0 = m^2 \sum_{f=1}^k (g_f \Sigma^{-1} \beta - \beta^{-1} \Sigma g_f), \quad (27)$$

$$\partial_+^2 A_0 = \lambda (\beta^{-1} i \partial_+ \beta). \quad (28)$$

Unlike the Schwinger model, where the equations of motion of the bosonised theory, (7) and (8), were given in terms of simple scalar fields, all the above fields are matrices. To obtain a set of solvable equations, we parametrise the matrix-valued fields and rewrite them as elements of the gauge group. That is,

$$g = e^{i\sigma_2 \varphi}, \quad \Sigma = e^{i\sigma_2 \eta}, \quad \beta = e^{i\sigma_2 \zeta}, \quad (29)$$

where the fields  $\varphi$ ,  $\eta$  and  $\zeta$  are scalars  $\sigma_2$  and is a generator of SU(2) group<sup>7</sup>. Rewriting the lagrangian (22) in terms of the above variables, introducing external colour probe charges  $q^a$ , with fixed colour charges  $a$ , and taking the static limit ( $\partial_0 = 0$ ), we obtain the equations of motion

$$\partial^2 \varphi = 8\pi m^2 \sin(\varphi + \eta + \zeta), \quad (30)$$

<sup>6</sup> We shall make frequent use of this limit and make parallels with the Schwinger model by taking the limit  $c_v \rightarrow 0$ .

<sup>7</sup> This simple parametrisation can be easily extended to SU(N).

$$\partial^2 \eta = \frac{-8\pi m^2}{(c_v + 1)} \sin(\varphi + \eta + \zeta), \quad (31)$$

$$\partial E = \partial^2 A_0 = \frac{(c_v + 1)}{2\pi} q^a (\delta(x - L/2) - \delta(x + L/2)) - \lambda \partial \zeta, \quad (32)$$

$$E = \frac{-1}{4\pi\lambda} \partial^2 \zeta + \frac{2m^2}{\lambda} \sin(\varphi + \eta + \zeta) - \quad (33)$$

The integration constant  $\alpha$  arising from the integration of equation (32) can be interpreted as the background electric field, i.e., as the theta vacuum;  $\theta = (2\pi\alpha/e)$  (see equations (9) and (10) of the Schwinger model for comparison.)

By making the substitution  $\phi = \varphi + \eta$ , the above four equations are reduced to the coupled equations,

$$\begin{aligned} \partial^2 \tilde{E} &= 4\pi\lambda^2 \tilde{E} + 8\pi m^2 \sin\left(\phi + \tilde{E} + \frac{\alpha}{\lambda}\right) \\ &\quad - 2\lambda q^a (c_v + 1) (T(x - L/2) - T(x + L/2)), \end{aligned} \quad (34)$$

$$\partial^2 \phi = \left(\frac{8\pi m^2 c_v}{c_v + 1}\right) \sin\left(\phi + \tilde{E} + \frac{\alpha}{\lambda}\right), \quad (35)$$

where  $\tilde{E} = \zeta - \alpha/\lambda$  and  $E = -\lambda \tilde{E} + \frac{c_v + 1}{2\pi} q^a (T - \bar{T})$  (similar to (11) of the Schwinger model).

### 3. MASSLESS QCD

For massless dynamical fermions, equation (34) simplifies to

$$\partial^2 \tilde{E} - 4\pi\lambda^2 \tilde{E} + 2q^a \lambda (c_v + 1) (T(x - L/2) - T(x + L/2)) = 0. \quad (36)$$

This equation is similar to the expression (12) of the Schwinger model and can be solved by the same techniques. The inter-charge potential, obtained in the fashion of the Schwinger model, is

$$V(L) = \frac{(c_v + 1)\sqrt{\pi} q^a L}{2e} \left(1 - e^{-\frac{(c_v + 1)eL}{\sqrt{\pi}}}\right). \quad (37)$$

Thus, massless QCD exhibits screening. It is worth mentioning that the screening potential of the Schwinger model (16) can be easily obtained by taking the limit  $c_v \rightarrow 0$ .

### 4. MASSIVE TWO-DIMENSIONAL QCD

The massive equations of motion, (34) and (35), are not exactly solvable. We first expand the sine term and solve the coupled equations for the field  $\phi$ . The quartic equation for the field  $\phi$  is

$$\partial^4 \phi - \left( \frac{8\pi m^2 c_v}{c_v + 1} + 4\pi\lambda^2 + 8\pi m^2 \right) \partial^2 \phi - \frac{32\pi^2 \lambda^2 m^2 c_v}{c_v + 1} \phi - \left( \frac{32\pi^2 \lambda c_v}{c_v + 1} \alpha + 16\pi m^2 c_v \lambda q^a (\mathbb{T} - \mathbb{T}) \right) = 0. \quad (38)$$

The electric field can be obtained from the solutions of the above equation by using expression (35) and the relation between  $E$  and  $\tilde{E}$ . Subsequently, the inter-charge potential is

$$V(L) = \frac{(c_v + 1)^2 q^a{}^2}{2} \times \left[ \left( \frac{4\pi\lambda^2 - m_+^2}{m_+^2 - m_-^2} \right) \left( \frac{1 - e^{-m_+ L}}{m_+} \right) + \left( \frac{m_+^2 - 4\pi\lambda^2}{m_+^2 - m_-^2} \right) \left( \frac{1 - e^{-m_- L}}{m_-} \right) \right], \quad (39)$$

where the mass scales  $m_{\pm}$ , arising from the solutions of the quartic equation (38), are given by

$$m_{\pm}^2 = 2\pi \left[ \lambda^2 + \left( 1 + \frac{c_v}{c_v + 1} \right) 2m^2 \right] \pm 2\pi \left[ \sqrt{\left( \lambda^2 + \left( 1 + \frac{c_v}{c_v + 1} \right) 2m^2 \right)^2 - 8 \frac{c_v}{c_v + 1} \lambda^2 m^2} \right] \quad (40)$$

The expression (39) for the inter-charge potential contains no confining term. Thus, massive QCD is in the screening phase. The confining potential of the massive Schwinger model is obtained by taking the limit  $c_v \rightarrow 0$ . In this limit, the mass scale  $m_-$  goes to zero and we recover expression (20) of the Schwinger model for  $\theta = 0$ . We also observe that, unlike the Schwinger model, the theta vacuum, i.e., the constant  $\alpha$  in (35), does not appear in the expression for the inter-charge potential. Thus, the phase structure of the non-abelian theory is not affected by the values of the theta angle.

## 5. CONCLUSION

By a simple semi-classical treatment of two dimensional QED and QCD, we have determined the phases of these theories. The introduction of the mass parameter in the Schwinger model causes a transition from the screening to the confining phase. This transition does not occur in the non-abelian theory where the screening phase prevails. It is worth mentioning that similar analysis were recently done in higher dimensions [2]. It has been shown that both massless and massive (for very large fermion masses) three-dimensional QED and in the screening phase [2].

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