# ABOUT THE MOTION AND THE EQUATIONS OF A POINT PARTICLE WITH SPIN (POINT STRUCTURAL PARTICLE)

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## I. INTRODUCTION

Starting with the work of Schrödinger [1], there have been many attempts (see for example the work [2]) to construct a classical theory of a point-spin particle. The main difficulty of these models [2], is that the notion of spin is introduced in mathematical formalisms together with other "hidden" variables which have no explicit physical meaning. Besides, in the models mentioned in [2], either the particles are not point objects, or the theory can be called classical only formally. In the papers [2, 3, 4] the theory of the point structure particle (p.s.p.) was developed, that is, a classical theory for a point particle with spin, and other variables that have a clear physical meaning.

In the present paper, after a brief review of the general theory of p.s.p. in stationary fields, we present the generalization for non stationary fileds, the spin-orbit interaction in the one body problem, the spin-spin and the spin-orbit interaction in the two bodies problem and the motion of a neutral p.s.p. in some concrete external stationary fields.

## II. PHYSICAL QUANTITIES AND EQUATIONS OF MOTION FOR A p.s.p. IN STATIONARY FIELDS

In works [2, 3, 4] the concept of p.s.p. as a cluster of point subparticles with masses $m_j$ charges $e_j$ coordinates $\vec{q}_j$, linear momentum $\vec{p}_j = m_j\dot{\vec{q}}_j$, was introduced. This cluster in the presence of external gravitational, electric and magnetic fields, with potentials $\phi(\vec{r})$, $\varphi(\vec{r})$, $\vec{A}(\vec{r})$ and respective intentisities:

$\vec{G} = -\vec{\nabla}\phi(\vec{r})$, $\vec{E} = -\vec{\nabla}\varphi(\vec{r})$ and $\vec{H} = \vec{\nabla}\times\vec{A}(\vec{r})$, is described by the system of equations:

$$m_j\ddot{q}_{ja} = m_j \cdot G_a(\vec{q}_j) + e_j E_a(\vec{q}_j) + \frac{e_j}{c} e_{\alpha\beta\gamma}\dot{q}_{j\beta}\cdot H_\gamma(\vec{q}_j) - \frac{\partial}{\partial q_{ja}} W(\cdots,\vec{q}_j - \vec{q}_k,\cdots), \quad \alpha = x, y, z \tag{1}$$

with the energy
$$E = \sum_j \left\{\frac{\vec{p}_j^{\,2}}{2m_j} + m_j\phi(\vec{q}_j) + e_j\varphi(\vec{q}_j)\right\} + W(\cdots,\vec{q}_j - \vec{q}_k,\cdots), \tag{2}$$

where W is the potential energy of interaction of the point subparticles, and it is related to the intensities of the external fields in such a way that the confinement condition is fulfilled, i.e.:

$$\left|\vec{q}_j(t) - \vec{q}_k(t)\right| \le l_o, \text{ if } \left|\vec{q}_j(t_o) - \vec{q}_k(t_o)\right| \le l_o, \tag{3}$$

for any j, k, $t_o$ and $t > t_o$, where $l_o$ is a small inaccessible, for the direct experimental observation interval of length.

Conditions (3) impose limitations to the intensities of the external fields and the structure of the confinement potential. As it will be seen, in certain conditions (for example for magnetic and gravitational perpendicular fields), the length of the p.s.p. growth up, and for a time $t = \tau_0$ condition (3) ceases to be fulfilled, and in this way it is possible to explain the disintegration of the p.s.p.

The condition (3), from the point of view of macroscopical observation, allows us to consider the p.s.p. as a point object with mass m, charge e, coodinate of the center of the mass $\vec{q}$ and linear momentum $\vec{p}$:

$$m = \sum m_j, \quad e = \sum e_j, \quad \vec{q} = \sum (m_j/m)\vec{q}, \text{ and } \vec{p} = \sum \vec{p}_j. \tag{4}$$

In papers [2, 3, 4] (for details see [4]) was shown that, with the help of the small parameters $l_o$ and $Z = \max Z_j$, $Z_j = e_j/m_j - e/m$, it is possible to obtain from (1) and (2) a closed system of equations for 15 independent variables that describe a p.s.p. as a whole, where appear 6 constants that also characterize the p.s.p.: the mass and the charge from (4), the giromagnetic factor g, the proper frequency $\omega_o$, and other two related with the former ones, $\kappa = \sum Z_j^2 m_j = lc^2 = 4mc^2(g-g_o)^2$ and; $g_o = e/2mc$ where the length $l_o$, in the case of the electron, coincides with the so called "classical radius". The 15 independent variables are: the coordinate and the momentum defined in (4), the dipole moment $\vec{d}$, the dipole speed $\vec{f} = \dot{\vec{d}}$, and the spin $\vec{S}$, defined as

$$\vec{d} = \sum e_j \vec{\xi}_j, \quad \vec{f} = \sum e_j \dot{\vec{\xi}}, \quad \vec{S} = \vec{s} - (\vec{d} \times \vec{f})/\kappa \text{ where } \vec{\xi}_j = \vec{q}_j - \vec{q}, \tag{5}$$

and $\vec{s} = \sum m_j \vec{\xi}_j \times \dot{\vec{\xi}}_j$ is the proper angular momentum. The system of equations are given by:

$$\dot{q}_\alpha = \frac{p_\alpha}{m}, \ \dot{p}_\alpha = mG_\alpha(\vec{q}) + eE_\alpha(\vec{q}) + \frac{e}{mc}\varepsilon_{\alpha\beta\gamma} p_\beta H_\gamma(\vec{q}) + \frac{\partial E_\alpha(\vec{q})}{\partial q_\beta} +$$

$$+ \frac{1}{2\kappa}\left(\frac{\partial^2 G_\alpha(\vec{q})}{\partial q_\beta \partial q_\gamma} + \frac{e}{m}\frac{\partial^2 E_\alpha(\vec{q})}{\partial q_\beta \partial q_\gamma}\right) d_\beta d_\gamma + \frac{1}{c}\varepsilon_{\alpha\beta\gamma}\left\{\left(\frac{p_\beta}{m} + \frac{e}{m\kappa} f_\beta\right)\frac{\partial H_\gamma(\vec{q})}{\partial q_\rho} d_\rho + \right. \tag{6}$$

$$\left. + \frac{e}{2m^2\kappa} p_\beta \frac{\partial^2 H_\gamma(\vec{q})}{\partial q_\rho \partial q_\mu} d_\rho d_\mu + f_\beta H_\gamma(\vec{q})\right\} + g\frac{\partial H_\beta(\vec{q})}{\partial q_\alpha} S_\beta + \frac{2c}{\kappa} g(g-g_o) S_\beta \frac{\partial^2 H_\beta(\vec{q})}{\partial q_\alpha \partial q_\gamma} d_\gamma$$

$$\dot{d}_\alpha = f_\alpha, \ \dot{f}_\alpha = -\omega_o^2 d_\alpha + \kappa E_\alpha(\vec{q},t) + \left(\frac{\partial G_\alpha(\vec{q},t)}{\partial q_\beta} + \frac{e}{m}\frac{\partial E_\alpha(\vec{q},t)}{\partial q_\beta}\right) d_\beta + \tag{7}$$

$$+ \varepsilon_{\alpha\beta\gamma}\left\{\left(\frac{\kappa}{mc} p_\beta + \frac{e}{mc} f_\beta\right) H_\gamma(\vec{q},t) + \frac{e}{m^2 c} p_\beta \frac{\partial H_\gamma(\vec{q},t)}{\partial q_\rho} d_\rho\right\} + 2c(g-g_o) S_\beta \frac{\partial H_\beta(\vec{q},t)}{\partial q_\alpha}$$

$$\dot{S}_\alpha = \varepsilon_{\alpha\beta\gamma} S_\beta \left\{ gH_\gamma(\vec{q},t) + \frac{2c}{\kappa} g(g-g_o)\frac{\partial H_\gamma(\vec{q},t)}{\partial q_\rho} d_\rho \right\} \tag{8}$$

and the energy (with accuracy of a constant terms) is:

$$H=\frac{p_\beta p_\beta}{2m}+m\phi+e\varphi+\frac{\partial\varphi}{\partial q_\beta}d_\beta+\frac{1}{2\kappa}\left(\frac{\partial^2\phi}{\partial q_\beta\partial q_\gamma}+\frac{e}{m}\frac{\partial^2\varphi}{\partial q_\beta\partial q_\gamma}\right)d_\beta d_\gamma-gS_\beta H_\beta-$$

$$-\frac{2c}{k}g(g-g_o)S_\beta\frac{\partial H_\beta}{\partial q_\gamma}d_\gamma+\frac{f_\beta f_\beta}{2\kappa}+\frac{\omega_o^2}{2\kappa}d_\beta d_\beta \tag{9}$$

It is easy to verify that the closed system of equations (6, 7, 8) with (9) lead exactly to the conservation law of the energy. It is also possible to verify that $\vec{S}$ = const. In the absence of magnetic fields, and $\vec{S}^2$ is an integral of motion in stationary fields. If we put $\vec{d}=\vec{f}=\vec{S}=0$, the calssical equations for a point particle (p.p.) are obtained. This means that our theory obeys the "correspondence principle". If we put $\vec{d}=\vec{f}=0$, with $S\neq 0$, the classical equations of the quantum Pauli theory are obtained, from which it can be seen that g is the giromagnetic factor and $\vec{S}$ is the spin. It is important to remark, that considering $\vec{d}=0$, means that $W=0$, and this is not in accordance with the confinement condition (3). In other words, this model requires that $d\neq 0$ if $\vec{S}\neq 0$.

## III. GENERALIZATION OF THE THEORY OF THE p.s.p. FOR NON STATIONARY FIELDS. THE SPIN-ORBIT AND SPIN-SPIN INTERACTIONS

For non stationary fields in the starting equations of motion (1) the intensities of the external fields will be time dependent. Repeating the previous procedure we obtain the closed system of equations (6, 7, 8), but with time dependent intensities of the external fields. For the variation of the energy in time is obtained the following expression:

$$\frac{d}{dt}\left\{\frac{p_\alpha p_\alpha}{2m}+\frac{f_\alpha f_\alpha}{2\kappa}-gS_\alpha H_\alpha-\frac{2c}{\kappa}g(g-g_o)S_\alpha\frac{\partial H_\alpha}{\partial q_\beta}d_\beta+\frac{\omega_o^2}{2\kappa}d_\alpha d_\alpha\right\}=$$

$$=\frac{p_\alpha}{m}\left\{mG_\alpha+eE_\alpha+\frac{\partial E_\alpha}{2q_\beta}d_\beta+\frac{1}{2\kappa}\frac{\partial^2\left(G_\alpha+\frac{e}{m}E_\alpha\right)}{\partial q_\beta\partial q_\gamma}d_\beta d_\gamma\right\}+f_\alpha\left\{E_\alpha+\frac{\partial\left(G_\alpha+\frac{e}{m}E_\alpha\right)}{\partial q_\beta}\frac{d_\beta}{\kappa}\right\} \tag{10}$$

Is easy to verify that: if we put $\vec{d}=\vec{f}=\vec{S}=0$ in (10) and in the equations of motion we obtain the equations for point particles in external non stationary fields. When the external fields do not depend on time, the equations of motion become the equations for the p.s.p. in external stationary fields, and from (10) is obtained the correspondent expression for the energy, with the correspondent law of conservation of the energy. For the spin are fulfilled all the considerations of the previous section.

To introduce the spin orbit interaction in the one body problem, we will consider that in the presence of an electric field with potential $\varphi(\vec{r})$, there is an additional specific interaction between the subparticles of the p.s.p., then the Lagrangian is:

$$L=\sum_j\left(\frac{m_j\dot{\vec{q}}_j\cdot\dot{\vec{q}}_j}{2}-e_j\varphi(\vec{q}_j)\right)-W-W',\ \text{where}:\ W'=\sum_j\frac{e_j}{2c^2}\left(\dot{q}_{j\beta}-\dot{q}_\beta-\frac{ef_\beta}{m\kappa}\right)\dot{q}_\beta\varphi(\vec{q}_j) \tag{11}$$

In this case is convenient to select as independent canonical variables:

$$\vec{q},\ \vec{P}=\sum_j \vec{P}_j,\ \vec{d},\ F_\beta=\sum_j Z_j P_{j\beta},\ \vec{S}=\sum_j \vec{q}_j\times\vec{P}_j-\vec{q}\times\vec{P}-\frac{\vec{d}\times\vec{F}}{\kappa};\quad \text{where}\quad P_{j\alpha}=\frac{\partial L}{\partial \dot{q}_{j\alpha}} \tag{12}$$

Repeating the procedure we arrive to a closed system of equations for these variables, with the Hamiltonian:

$$H=\frac{P_\beta P_\beta}{2m}+\frac{F_\alpha F_\alpha}{2\kappa}+\frac{\omega_0^2}{2\kappa}d_\beta d_\beta+e\varphi+d_\beta\frac{\partial\varphi}{\partial q_\beta}+\frac{e}{m}\frac{d_\beta d_\gamma}{2\kappa}\frac{\partial^2\varphi}{\partial q_\beta\partial q_\gamma}+\frac{1}{2c^2}\frac{P_\alpha}{m}\left[\left(1-\frac{e^2}{m\kappa}\right)F_\alpha\varphi+\right.$$

$$\left.+cg\varepsilon_{\gamma\beta\alpha}S_\gamma\frac{\partial\varphi}{\partial q_\beta}+2c^2\left(g(g-g_0)\varepsilon_{\nu\gamma\alpha}S_\nu-g_0^2\frac{d_\gamma F_\alpha}{\kappa}\right)\frac{d_\beta}{\kappa}\frac{\partial^2\varphi}{\partial q_\beta\partial q_\gamma}\right]. \tag{13}$$

The term $H_{SO}=\frac{gc}{2mc^2}\vec{S}\cdot\left[\vec{\nabla}\varphi\times\vec{P}\right]$ for a Coulomb potential $\varphi=\frac{Z|e|}{r}$, i.e., $H_{SO}=-\frac{g|e|Z}{2mcr^3}\vec{S}\cdot\vec{L}$ is the spin orbit interaction. This result is valid not only for electrons but also for other particles and nucleons, as g is a proper characteristic of the p.s.p.

We can also check the conservation laws of the energy and the square of the spin.

In the two bodies problem we have two clusters of classical point subparticles with masses $m_{1j}, m_{2k}$, charges $e_{1j}, e_{2k}$, coordinates $\vec{q}_{1j}, \vec{q}_{2k}$, and linear momenta $\vec{p}_{1j}=m_{1j}\dot{\vec{q}}_{1j}, \vec{p}_{2k}=m_{2k}\dot{\vec{q}}_{2k}$, that are at such a distance, that their confinement potentials $W_1(\cdots,\vec{q}_{1j}-\vec{q}_{1j'},\cdots)$ and $W_2(\cdots,\vec{q}_{2k}-\vec{q}_{2k'},\cdots)$ act only over the subparticles of each p.s.p., i.e., the coordinates of the subparticles of each p.s.p. obey the relations (for any j, j´, k, k´and t)

$$\left|\vec{q}_{1j}(t)-\vec{q}_{1j'}(t)\right|\le l_{o1},\ \left|\vec{q}_{2k}(t)-\vec{q}_{2k'}(t)\right|\le l_{o2},\ \text{and}\ \left|\vec{q}_{1j}(t)-\vec{q}_{2k}(t)\right|>l_{o1}+l_{o2},$$

where $l_{o1}$ and $l_{o2}$, are small inaccesible, for the direct observation, intervals of lenth.

The electromagnetic interaction between the subparticles of each p.s.p., considering up to terms of order $(v/c)^2$, will be considered with the help of the Darwing Lagrangian [5].

As in the previous case, we select for each p.s.p. the independent variables given by (12), and repeating the previous procedure we arrive at a closed system of 30 equations of motion that is consistent with the expression of the energy. In that expression appears the terms of the Breit interaction [6] of quantum theory:

$$U_{SO}=\frac{e_1 g_2}{m_1 c r^3}\left(\vec{S}_2\vec{L}_1\right)+\frac{e_2 g_1}{m_2 c r^3}\left(\vec{S}_1\vec{L}_2\right)\quad\text{and}\quad U_{SS}=g_1 g_2\left[\frac{\left(\vec{S}_1\vec{L}_2\right)}{r^3}-\frac{3\left(\vec{r}\vec{S}_1\right)\left(\vec{r}\vec{S}_2\right)}{r^5}\right]$$

where $\vec{L}_1=\left(\vec{r}\vec{P}_1\right)$ and $\vec{L}_2=\left(\vec{r}\vec{P}_2\right)$. The terms of $U_{SO}$ are the classical representation of the spin-orbit interactions between both p.s.p. and those of $U_{SS}$ are the classical representations of the spin-spin interactions between the p.s.p., we have proved that the classical model of the p.s.p. in also able to give a classical explanation of the spin orbit and spin-spin interactions between two particles.

This Hamiltonian conduces exactly to the conservation laws of: the energy, the total momentum, the total angular momentum and the square of the spin for each p.s.p.

## IV. ABOUT THE MOTION OF A NEUTRAL P.S.P. IN SOME STATIONARY EXTERNAL FIELDS

The trajectory $\vec{q}(t)$ and the values of other variables, $\vec{d}(t), \vec{S}(t),\cdots$, for instants $t > 0$, are uniquely determined by the motion equations (6, 7, 8) after the known values $\vec{q}_0, \vec{p}_0, \vec{d}_0, \vec{f}_0, \vec{S}_0$ of the independent variables at t = 0. We will consider a neutral p.s.p. ($e = g_0 = 0$). It is convenient to introduce instead of the variables $\vec{d}$ and $\vec{f}$ other physical quantities with dimensions of length and linear momentum: $\vec{\xi} = \vec{d}/\sqrt{\kappa m}$, $\vec{\eta} = m\dot{\vec{\xi}} = \sqrt{m/\kappa}\,\vec{f}$, and it is possible to show that $\vec{\xi}$ for neutral P.S.P. is proportional to the vector that joints the center of positive charge with the negative one. This means that if $\vec{\xi}$ grow up with time the condition (3) may not fulfilled and the p.s.p. desintegrates.

In the absence of external fields (free motion), the solutions of the equations (6, 7, 8) are:

$$q_\alpha = q_\alpha^0 + \dot{q}_\alpha^0 t,\ \xi_\alpha = \xi_\alpha^0 \cos\omega_0 t + \dot{\xi}_\alpha^0/\omega_0 \operatorname{sen}\omega_0 t, \text{ and } S_\alpha = S_0 \tag{14}$$

from which it can be seen that the trajectory of a free neutral p.s.p. is the same as for a free p.s., the proper frequency $\omega_0$, is the frequency of the oscillations (rotations) of the dipole moment, and the projections of the spin are constant. The mean value in time of the dipole moment is equal to zero, using the definition:

$$\langle d_\alpha \rangle = 1/T \int_0^T \sqrt{\kappa m}\,\xi_\alpha(t)dt = 0 \quad \text{where} \quad T = 2\pi/\omega_0 \tag{15}$$

In classical or quantum theory of p.p. we have no physical characteristic analogous to the proper frequency. Only in the work of Schrödinger [1] for the "Ziterbewegung" of a free quantum relativistic spin particle does such a quantity appears with the value $\omega_0 \sim m_0c^2/\hbar$. It is important to remark that the value of $\omega_0$ according to [1] is very large, for example for a neutron it is of the order of $10^{24}$ $s^{-1}$, this means that for a reasonable period of time $\Delta t < 10^{-12}$s, the dipole moment makes more than $10^{12}$ oscillations (or rotations), and in a direct measurement, one measure not $\vec{d}$ but $\langle\vec{d}\rangle$ according to (15).

This means that we need to estimate the possible values of $\vec{\xi}^0$ and $\dot{\vec{\xi}}^0$. For this purpose, if we accept according to [1] that $\omega_0 = 2mc^2/3\hbar$ and that the internal energy is the relativistic rest energy $mc^2$, with the help of the virial theorem for the mean value of the internal kinetic and potential energies, we arrive to:

$-c < \dot{\xi}_\alpha^0 < c,\quad -3/2\lambda_c < \xi_\alpha^0 < 3/2\lambda_c,\ \lambda_c = \hbar/mc$, where $\lambda_c$ is the Compton wave length.

In a homogeneus gravitational field, $\vec{G}$ = const, it is possible to verify that the trajectory of the p.s.p. is the same as for a p.p., and the solutions for the dipole moment and the spin are the same as for a free p.s.p.

In a homogeneus electric field $\vec{E}$ = const, the trajectory of the p.s.p. is again the same that for a p.p. The spin $\vec{S}$ = const. But the mean value of the dipole momentum is: $\langle\vec{d}\rangle = \kappa\vec{E}/\omega_0^2$ which means that $\kappa = lc^2$ determines the polarizability of the neutral p.s.p. This give a value of 05 × $10^{-3}$ $fm^3$ for the polarizability of the neutron.

In a homogeneus magnetic $\vec{H}$ = const, also takes place a polarization of the p.s.p. and its trajectory may differ from the trajectory of the p.p. only for stremly low velocities.

The motion in perpendicular homogeneous gravitational and magnetic field $\vec{G} = (G,0,0), \vec{H} = (0,0,H)$; in the solutions for the dipole momentum appears a term $vGt/\omega_1^2$ that increases monotonically with time, where:

$\omega_1^2 = \omega_0^2 + v^2$, $v = \sqrt{\kappa/m}\,H/c$, and for long intervals of time the p.s.p. may disintegrate. For the deflection angle, when $\omega_0^2 >> v^2$, and neglecting the oscillating term, we obtain

$\theta = \operatorname{arctg}\left\{\frac{\dot{q}_y(t)}{\dot{q}_x(t)}\right\} = \operatorname{arctg}\left\{v\xi_x^0 \Big/ \left(\dot{q}_x^0 + Gt - v\xi_y^0\right)\right\}$ from where it is seen that if $G < 0$, the velocities in the denominator decrease, and the deflection angle increase.

In the case of the motion in a Coulomb field $\varphi = Z|e|/r$ we are able to calculate [7] the cross section of incidence and obtain the value $\sigma_{inc} = 2\pi[\delta/E_0]^{1/2}$, where $E_0 = m\dot{q}_0^2/2$ (the initial energy), $\delta = Z^2e^2\kappa/2\omega_0^2$. This is the same dependence of $E_0$ that predicts the quantum Breit-Wigner formulas.

From the examples we have studied of the motion of a neutral p.s.p. in stationary gravitational, electric and magnetic homogeneous fields and non homogeneous electric field, it can be seen that the motion of the neutral p.s.p. may differ from the motion of p.p. only for very low velocities.

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