CHAOS AND NON-ARCHIMEDEAN METRIC IN THE BERNOULLI MAP

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ABSTRACT
Ultrametric concepts are applied to the Bernoulli Map, showing the adequateness of the non-Archimedean metrics to describe in a simple and direct way the chaotic properties of this map. Lyapunov exponent and Kolmogorov entropy appear to find a better understanding. A p-adic time emerges as a natural consequence of the ultrametric properties of the map.

RESUMEN
Se aplican los conceptos de la geometría ultramétrica al mapa de Bernoulli, demostrando que la métrica no Arquimedeana es la adecuada para describir de forma simple y directa las propiedades caóticas de este mapa. En este contexto, tanto el exponente de Lyapunov como la entropía de Kolmogorov encuentran un contenido conceptual claro. Como consecuencia de las propiedades ultramétricas del mapa, emergen también propiedades p-adicas del tiempo en este contexto.

INTRODUCTION

After the work by Mézard **et al**. [1], ultrametricity has triggered the interest in a wide range of physical phenomena, due to its applications in different topics: spin glasses, mean field theory, turbulence, nuclear physics. Also optimization theory, evolution, taxonomy, protein folding benefits from it (for an excellent review see references 2-5). Wherever a hierarchical concept appears, non-Archimedean analysis is an adequate tool to study the problem.

Ultrametricity is a promising tool in the theory of branching processes, which, at the same time, has revealed its possibilities in the study of self-organized critical processes [6-8]. It seems possible to find simpler tools to describe the geometry of these processes. Here, we illustrate the advantages of a hierarchical representation in the case of the Bernoulli shift. This will permit, using simple geometric considerations, to determine the magnitudes governing the system, and the advantages of a p-adic metric will be stressed over the Euclidean one. The ultrametric distance will be shown to be consistent with the characteristic behavior of this chaotic unidimensional map.

Since in this paper we explore the application of ultrametricity to link the Bernoulli map with a branching structure, this will reveal the possibilities of assign an ultrametric measure to processes that, apparently, are not linked with a given metric (e.g. minority game [9] and related problems) so that an adequate understanding of the ultrametric properties of a given process may lead to its deeper understanding.

In ultrametric spaces, concepts such as exponential separation of neighboring trajectories, and characteristic parameters (Lyapunov exponents and Kolmogorov entropy) seem to find a simpler understanding than with the Euclidean metric.

As an example, where Euclidean metric is not very adequate, let us consider the Baker's map [10]. The interval $[0,1]\times[0,1]$ is mapped to $[0,1]\times[0,1]$. Therefore, the distance between two points can't be larger than the distance between two opposite corners in $[0,1]\times[0,1]$.

Nonetheless, the Baker's map has got a Lyapunov exponent bigger than one. Then, the distance between neighboring points grows exponentially in a finite region of the phase space. In the Euclidean space we would have to define the distance in this case as the Euclidean length of the shortest path lying entirely within the region that has suffered the deformation [11]. As any nontrivial norm is equivalent to the Euclidean or any of the p-adics (Ostrowski's theorem [12]), it would be convenient to measure the distance between points in the Baker's map with a p-adic metric.

An ultrametric space is a space endowed with an ultrametric distance, defined as a distance satisfying the inequality

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$$d(A,C) \leq Max\{d(A,B), d(B,C)\} \quad (1)$$

(A,B and C are points of this ultrametric space), instead of the usual triangular inequality, characteristic of Euclidean geometry

$$d(A,C) \leq d(A,B) + d(B,C) \quad (2)$$

A metric space **E** is a space for which a distance function $d(x,y)$ is defined for any pair of elements (x,y) belonging to **E**.

A norm satisfying

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad (3)$$

is called a non-Archimedean metric, because equation (3) implies that

$$\|x+x\| \leq \|x\| \quad (4)$$

holds, and equation (4) does not satisfy the Archimedes principle:

$$\|x+x\| \geq \|x\| \quad (5)$$

A metric is called non-Archimedean or ultrametric, if (1) holds for any three points (x,y,z).

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad (6)$$

A non-Archimedean norm induces a non-Archimedean metric:

$$d(x,z) = \|x-z\| \leq \max\{d(x,y), d(y,z)\} \quad (7)$$

Equation (7) implies a lot of surprising facts, e.g., that all triangles are isosceles or equilateral and every point inside a ball is itself at the center of the ball, furthermore the diameter of the ball is equal to its radius.

An example of ultrametric distance is given by the p-adic distance, defined as:

$$d_p(x,y) = \|x-y\|_p \quad (8)$$

where the notation defines the p-adic absolute value:

$$\|x\|_p \equiv p^{-r} \quad (9)$$

where p is a fixed prime number , $x \neq 0$ is any integer, and r is the highest power of p dividing x. Two numbers are p-adically closer as long as r is higher, such that p^r divides $\|x-y\|$. Amazingly, for p = 5 the result is that 135 is closer to 10 than 35.

Any positive or negative integer can be represented by a sum

$$x = \sum_{i=0}^{\infty} a_i p^i \quad (10)$$

where

$$0 \leq a_i \leq p-1 \quad (11)$$

If negative exponents are considered in the sum, rational numbers can also be represented. Such a representation is unique. The set of all sums Q_p is the field of p-adic numbers, and contains the field of rational numbers Q but is different from it.

Lyapunov exponent and Kolmogorov entropy

With the above description the p-adic numbers have a hierarchical structure, whose natural representation is a tree. Let us now use this description to work with the Bernoulli map (See [10]):

$$\begin{aligned} x_{n+1} &= 2x_n \bmod 1 \\ n &= 0,1,2... \end{aligned} \quad (12)$$

Here, we may note that the numbers can be represented as a set of points in a straight line or by a hierarchical structure, depending on the definition of distance (Euclidean or Archimedean) as we see below:

Let us represent the initial value (state) to be mapped into the unit interval by the sequence $0, a_1.....a_N.....$ with $a_i =$ 0 or 1 to denote the initial value in binary notation.

It is possible to reorder these sequences as a hierarchical tree. To get it, let us do the following process to represent the result of the application of the Bernoulli map:

We begin at an arbitrary point. We read, consecutively, the values of a_i, from i = 1 to N, of the sequence $a_1 \ldots a_N \ldots$. When a_i takes the value 0 we move to the left, and the same distance down. When a_i takes the value 1 we do the same, but moving on the right. The result is 2^N branches of a hierarchical tree. Any finite path inside this branching structure represents univocally a possible finite sequence $a_1 \ldots a_N \ldots$.

Thus, for instance, the sequence 0,0110 represent: left, right, right, left.

The distance $d(x_i, x_j)$ between two branches (sequences) x_i, x_j in this tree is given by

$$d(x_i, x_j) = \begin{cases} 2^{(m-n)} \rightarrow i \neq j \\ 0 \rightarrow i = j \end{cases} \quad (13)$$

where m is the number of levels one must move up the tree to find a common branch linking x_i and x_j, and N is the number of levels (the length of the sequence). This is equivalent to

$$d(x_i, x_j) = \begin{cases} 2^{-h} \rightarrow i \neq j \\ 0 \rightarrow i = j \end{cases} \quad (13a)$$

where h is the position of the last block a_h in which a_i (i = 1,....,h) are common to the two sequences x_i, x_j. It means that the numbers x_i and x_j are close up to the h^{th} binary place. This distance is an ultrametric one.

To calculate the Lyapunov exponent it is necessary to know how neighboring points $x_0 + \varepsilon$ and x_0 evolve during the Bernoulli map. Let ε be equal to $2^{-h}[1 + 2^{-\delta_1} + 2^{-\delta_2} + \ldots] > 2^{-N}$, then the first different position between $x_0 = 0$, $a_1a_2\ldots a_{h-1}a_N\ldots$ and $x_0 + \varepsilon$ is a_h.

Then, it is necessary to move up the tree N - h +1 levels from the bottom line to find the common branch in the position a_{h-1}(obviously, the last common figure between x_0 and $x_0 + \varepsilon$). So,

$$d(x_0 + \varepsilon, x_0) = 2^{-(h+1)} \quad (14)$$

and

$$d(f^n(x_0 + \varepsilon), f^n(x_0)) = 2^{-h+1+n} \quad (15)$$

because the iteration f^n moves away the common branch n positions from the bottom level.

To calculate the Lyapunov exponent it is necessary to express the exponential growth of the distance between two neighboring points:

$$\lim_{n\to\infty} \lim_{n\to\infty} 2^{\lambda n}\varepsilon = \lim_{n\to\infty} \lim_{\varepsilon\to 0} d(f^n(x_0 + \varepsilon), f^n(x_0)) \quad (16)$$

Since the base for measuring the p-adic distance in our space is the number 2, in the preceding equation we have expressed the exponential growth with $2^{\lambda n}$ instead of $e^{\lambda n}$.

Replacing and $d(f^n(x_0 + \varepsilon), f^n(x_0))$ in the preceding equation we obtain

$$\lim_{n\to\infty} \lim_{h\to\infty} 2^{-h}(1 + 2^{-\delta_1} + 2^{-\delta_2} + \mathrm{K})2^{\lambda n} = \lim_{n\to\infty} \lim_{h\to\infty} 2^{-h+1+n}, \quad (17)$$

from (17) it can be easily observed that. Since the Lyapunov exponent in the Bernoulli map is ln2 [7], we recover this result with p-adic metric, since $2 = e^{\ln 2}$. It means that each unit time interval implies a new doubling of branches in each node of the hierarchical tree. Then, once a unit time interval has elapsed, the number of levels one must move up the tree to find a common branch increases in one. This result will be crucial to understand how the information is lost in the course of time.

In unidimensional maps, as the one considered here, the Kolmogorov entropy coincides with the Lyapunov exponent [7]. The Kolmogorov entropy expression is:

$$K = \lim_{n\to\infty} \lim_{\tau\to 0} \frac{1}{n\tau} \sum_{i_1 \mathrm{K}\, i_n} p_{i_1 \mathrm{K}\, i_n} \lg_2 p_{i_1 \mathrm{K}\, i_n} \quad (18)$$

where $p_{i_1 \mathrm{K}\, i_n}$ is the probability to reach the i_n state of the system in the phase space following a given path $i_1i_2\ldots i_n$. It can be seen that in our case this probability only depends on the final state i_n because for each state there is just one path, i.e., that given by the sequence $i_1i_2\ldots i_n$. Besides, the number of states in the n^{th} level is 2^n, and τ is the time elapsed to pass from one state to a successive one. The probability to occupy one of the 2^n states is $p_n = p_{i_1 i_2 \mathrm{K}\, i_n} = \frac{1}{2^n}$ and it results

$$K = \lim_{n\to\infty} \lim_{\tau\to 0} \frac{2}{\tau 2^n} \quad (19)$$

But the distance between two successive states of the n^{th} level is 2^{1-n}, because they are common until the $(n-1)^{th}$ level. Since the speed v to pass from one sequence to the next is constant in Bernoulli map, i.e., $\nu = \frac{2^{1-n}}{\tau} = 1$ the time τ elapsed between these two successive states is k = 1. As expected k = 1, coinciding with the Lyapunov exponent. Notice that the existence of a p-adic proper time is essential for the coincidence of the Kolmogorov entropy and the Lyapunov exponent. The spatial p-adic structure is unavoidable joined to the p-adic structure of proper time.

Therefore, we can say that this problem is endowed with a p-adic spatial and temporal geometry instead of a sole p-adic spatial geometry. To see the importance of the introduction of a p-adic time, see [13].

The Kolmogorov entropy measures the loss of information in the process. From our representation this loss of information can be easily seen, since the process of separation of trajectories is such that for any step the increase of the distance between two points duplicates the number of branches through which this increment can be reached. We are loosing information because we don' t know exactly the way we are separating two states.

On the other hand, we can see that in the ultrametric space the natural time of the system is also ultrametric. The time of transition between two sequences x_i, x_j satisfies the same expression (13) as the distance between x_i, x_j.

Besides, subsequent behavior of two states that separate in a given point in the ultrametric space depends of the point in which this separation occurs, revealing that ultrametricity can be applied to processes of decision (like minority games, aging effects, hierarchical processes, etc), all these fields in which ultrametric concepts have been poorly applied. The application of ultrametricity to the minority game will be treated in future works.

CONCLUSIONS

It was verified that the Bernoulli map leads to a hierarchical structure in the p-adic metric. With the ultrametric distance the Lyapunov exponent and the Kolmogorov entropy acquire a better understanding and a direct geometric interpretation is supplied by the hierarchical structure. The p-adic metric seems to be the natural metric of this map. The hierarchical structure generates p-adic properties for the temporal evolution.

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