FAST COMPUTATION OF HYDRODYNAMIC GREEN'S FUNCTION

CÁLCULO RÁPIDO DE LA FUNCIÓN DE GREEN HIDRODINÁMICA

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A fast and accurate numerical method for computing the hydrodynamic Green's function will be presented. The method is based on writing the hydrodynamic Green's function in terms of an auxiliary analytic function which is computed by solving a boundary integral equation with the generalized Neumann kernel. Application of the presented method to compute the stream function of fluid motion due to a finite number of point vortices in an incompressible fluid on planar multiply connected domains of high connectivity with complicated boundaries will be presented.

Será presentado un método rápido y preciso para calcular la función de Green hidrodinámica. El método se basa en escribir la función de Green hidrodinámica en términos de una función analítica auxiliar, que es calculada resolviendo una ecuación integral con condiciones de frontera usando el kernel de Neumann generalizado. Se montara una una aplicación del presente método para calcular la función de flujo para el movimiento del fluido debido a un numero finito de vórtices puntiformes en un fluido incompresible, sobre dominios planos multiplemente conexos de alta conectividad, con fronteras complicadas.

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I INTRODUCTION

Suppose that R is a multiply connected domain in the extended complex plane $\mathbb{C} \cup \{\infty\}$. The domain R can be bounded of connectivity $m+1$ or unbounded of connectivity m. For bounded R, the boundary $\Gamma = \partial R$ consists of $m+1$ closed smooth Jordan curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ where the curve Γ_0 enclose the other curves $\Gamma_1, \ldots, \Gamma_m$ (see Figure 1). For unbounded R, $\Gamma = \partial R$ consists of m closed smooth Jordan curves $\Gamma_1, \ldots, \Gamma_m$ and $\infty \in G$ (see Figure 2). The orientation of Γ is such that R is always on the left of Γ.

Lin [1, 2] in 1941 considered a special Green function for multiply connected domains in the complex plane which was named by Flucher & Gustafsson [3] as *the hydrodynamic Green's function* (see also [3–5]). Since then, the hydrodynamic Green's function plays a key role in solving several fluid dynamic problems in multiply connected domains in the complex plane (see e.g., [3–7]).

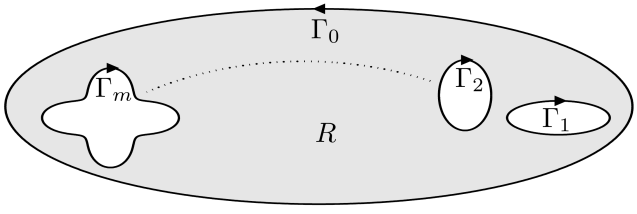

Figure 1. A bounded multiply connected domain R of connectivity $m+1$.

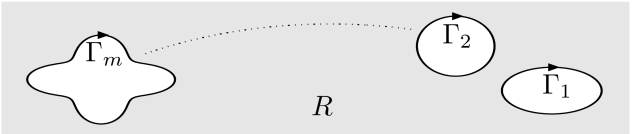

Figure 2. An unbounded multiply connected domain R of connectivity m.

The hydrodynamic Green's function $G(x, y; x_0, y_0)$ with respect to the two points (x, y) and (x_0, y_0) in the domain R is defined as in the following definition from [1] (see also [3–5]). In this paper, for simplicity, we shall write $G(z; z_0)$ instead of $G(x, y; x_0, y_0)$ where $z = x + \mathrm{i}y$ and $z_0 = x_0 + \mathrm{i}y_0$.

Definition 1 *The hydrodynamic Green's function $G(z; z_0)$ is defined with respect to two points $z = x + \mathrm{i}y$ and $z_0 = x_0 + \mathrm{i}y_0$ in the domain R by the following three conditions.*

1. *The function*

$$g(z; z_0) = -G(z; z_0) - \frac{1}{2\pi} \log |z - z_0| \tag{1}$$

is harmonic with respect to (x, y) throughout the domain R including at the point (x_0, y_0).

2. *If $\frac{\partial G}{\partial \mathbf{n}}$ denotes the normal derivative of G on a curve (with $z = x + \mathrm{i}y$ as a variable) and ds denotes an element of arc, then*

$$G(z; z_0) = c_k \text{ over } \Gamma_k, \; k = 1, 2, \ldots, m, \tag{2a}$$

$$\int_{\Gamma_k} \frac{\partial G}{\partial \mathbf{n}} ds = 0, \qquad k = 1, 2, \ldots, m, \tag{2b}$$

where $c_1, c_2, \ldots, c_m$ are real constants.

3. *(a) If G is bounded, then*

$$G(z; z_0) = 0 \quad \text{over } \Gamma_0. \tag{3a}$$

(b) If R is unbounded, then over a very large circle of radius

r_0, the function G behaves as follows

$$G(z;z_0) = -\frac{1}{2\pi}\log r_0 + O\left(\frac{1}{r_0}\right), \tag{3b}$$

$$\frac{\partial G}{\partial s} = O\left(\frac{1}{r_0^2}\right), \tag{3c}$$

$$\frac{\partial G}{\partial \mathbf{n}} = -\frac{1}{2\pi r_0} + O\left(\frac{1}{r_0^2}\right), \tag{3d}$$

where $\frac{\partial G}{\partial s}$ is the tangential derivative along the circle.

Theorem 1 (Lin [1]) *The function $G(z;z_0)$ defined by the conditions in Definition 1 exists uniquely and satisfies the reciprocity property*

$$G(z;z_0) = G(z_0;z). \tag{4}$$

An explicit representation of the hydrodynamic Green's function $G(z;z_0)$ for multiply connected circular domains has been given in [4, 6, 8, 9]. This explicit representation is described in terms of the Schottky-Klein prime function associated with circular domains [10]. If the conformal mapping from the general multiply connected domain R onto a circular domain is known, then the method presented in [4, 6] can be used to calculate the hydrodynamic Green's function in the domain R. However, computing the conformal mapping from the general multiply connected domain R onto a circular domain is not easy. In fact, it is more difficult than computing the hydrodynamic Green's function in the original domain R (see e.g., [11–13]). A boundary integral method for computing the function $G(z;z_0)$ for the coastline multiply connected domain has been presented in [7]. The method is based on computing the conformal mapping from the original coastline domain onto the unit disk with circular slits. The hydrodynamic Green's function in the domain R is then computed from the conformal mapping.

This paper presents a new numerical method for computing the hydrodynamic Green's function $G(z;z_0)$ for general bounded or unbounded multiply connected domains R. The method is based on using a boundary integral equation with the generalized Neumann kernel to compute $G(z;z_0)$ in the domain R without using conformal mappings. The presented method has the ability to compute the hydrodynamic Green's function for domains with high connectivity, domains with complex geometry, domains with close-to-touching boundaries, and domains with piecewise smooth boundaries. As an application, we use the presented method to compute the stream function of fluid motion due to a finite number of point vortices in an incompressible fluid on planar multiply connected domains including a real-world problem domain.

In the method presented in [7], both the kernel and the right-hand side of the integral equation depends on z_0. While, in the method presented in this paper, only the right-hand side of the integral equation depends on z_0. This is a very important advantage of the presented method. For example, to compute the hydrodynamic Green's function $G(z;z_i)$ for l pints z_i, $i = 1, 2, \ldots, l$, using the method presented in [7], we have to solve l different integral equations. However, for the presented method, we need to solve only one integral equation but with l different right-hand sides. If a proper method for solving linear systems with multiple right-hand sides is used to solve the discretized linear system of the integral equation (see e.g., [14]), then the computational cost of the presented method will be much less than the computational cost of the method presented in [7].

Another important Green's function is *the classical Green's function* which is defined differently from the hydrodynamic Green's function. Crowdy & Marshall [8] called the hydrodynamic Green's function as the modified Green's function and the classical Green's function as the first-type Green's function (see also [9]). For a numerical method based on the integral equation with the generalized Neumann kernel for computing the classical Green's function on doubly connected domains, we refer the reader to [15, 16].

II THE GENERALIZED NEUMANN KERNEL

In this section, we review the definition of the generalized Neumann kernel on multiply connected domains. For more details on the generalized Neumann kernel, we refer the reader to [17–23].

The curve Γ_j is parameterized by a 2π-periodic complex-valued function $\eta_j(t)$ with $t \in J_j = [0, 2\pi]$. The total parameter domain J is the disjoint union of the intervals J_0 (for bounded R), $J_1, \ldots, J_m$. We define a parameterization of the whole boundary Γ as the complex function $\eta(t)$ defined on J by

$$\eta(t) = \begin{cases} \eta_0(t), & t \in J_0 \ \text{(for bounded } R), \\ \eta_1(t), & t \in J_1, \\ \quad\vdots & \\ \eta_m(t), & t \in J_m. \end{cases} \tag{5}$$

Any real-valued or complex-valued function $\phi(\eta)$ defined on the boundary Γ can be interpreted via $\hat{\phi}(t) := \phi(\eta(t))$ as a 2π-periodic function of the parameter t on J, and vice versa. So, in this paper, we shall not distinguish between $\phi(t)$ and $\phi(\eta(t))$. If $h(t)$ is a piecewise constant function defined on J by

$$h(t) = h_j \quad \text{for} \quad t \in J_j$$

where h_j is a real constant for $j = 0$ (for bounded R),1,2,...,m, then for simplicity, the function $h(t)$ will be written as $h = (h_0, h_1, h_2, \ldots, h_m)$ for bounded R and $h = (h_1, h_2, \ldots, h_m)$ for unbounded R.

We define a complex-valued function A on Γ by

$$A(t) = \begin{cases} \eta(t) - \alpha, & \text{for bounded } R, \\ 1, & \text{for unbounded } R. \end{cases} \tag{6}$$

The generalized Neumann kernel formed with A and η is defined by [17,18]

$$N(s,t) = \frac{1}{\pi}\,\mathrm{Im}\left(\frac{A(s)}{A(t)}\,\frac{\eta'(t)}{\eta(t)-\eta(s)}\right). \tag{7}$$

We define also a kernel

$$M(s,t) = \frac{1}{\pi}\,\mathrm{Re}\left(\frac{A(s)}{A(t)}\,\frac{\eta'(t)}{\eta(t)-\eta(s)}\right). \tag{8}$$

To ensure the continuity of the generalized Neumann kernel, we assume that the boundary Γ is a C^2 smooth curve, i.e., we assume that $\eta''(t)$ is continuous and $\eta'(t) \neq 0$ on J. However, as explained in [19, 22], the method presented in this paper can be used even for domains with piecewise smooth boundaries.

Lemma 1 ([18]) *(a) The kernel N is continuous with*

$$N(t,t) = \frac{1}{\pi}\left(\frac{1}{2}\,\mathrm{Im}\,\frac{\eta''(t)}{\eta'(t)} - \mathrm{Im}\,\frac{A'(t)}{A(t)}\right). \tag{9}$$

(b) When $s,t \in J_j$ are in the same parameter interval J_j, then

$$M(s,t) = -\frac{1}{2\pi}\cot\frac{s-t}{2} + M_1(s,t) \tag{10}$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(t,t) = \frac{1}{\pi}\left(\frac{1}{2}\,\mathrm{Re}\,\frac{\eta''(t)}{\eta'(t)} - \mathrm{Re}\,\frac{A'(t)}{A(t)}\right). \tag{11}$$

Thus, the integral operator

$$\mathbf{N}\mu(s) = \int_J N(s,t)\mu(t)dt \tag{12}$$

is a Fredholm integral operator and the operator

$$\mathbf{M}\mu(s) = \int_J N(s,t)\mu(t)dt \tag{13}$$

is a singular integral operator.

Theorem 2 ([24]) *If λ is an eigenvalue of $\mathbf{N}$, then $\lambda \in [-1,1)$.*

III AN INTEGRAL EQUATION FOR THE HYDRODYNAMIC GREEN'S FUNCTION

This section presents a boundary integral equation for computing the hydrodynamic Green's function $G(z;z_0)$ for bounded and unbounded multiply connected domains R. The kernel of the integral equation is the generalized Neumann kernel discussed in the previous section and the right-hand side of the integral equation depends on the point z_0. By solving the integral equation, we obtain the boundary values of an axillary analytic function f. The values $f(z)$ for interior points $z \in R$ can be computed by the Cauchy integral formula. By obtaining the values of the axillary function $f(z)$, we obtain the values of the hydrodynamic Green's function $G(z;z_0)$. The numerical implementation of the integral equation will be given in the forthcoming sections.

Suppose that a point z_0 is fixed in R and the function γ is defined by

$$\gamma(t) = \frac{1}{2\pi}\log|\eta(t) - z_0|. \tag{14}$$

Then, we have the following theorem from [20,21].

Theorem 3 *For the function γ given by (14), there exists a unique piecewise constant function $h = (h_0, h_1, h_2, \ldots, h_m)$ and a unique 2π-periodic Hölder continuous function μ such that*

$$A(t)f(\eta(t)) = \gamma(t) + h(t) + \mathrm{i}\mu(t), \quad t \in J, \tag{15}$$

are boundary values of an analytic function f in R with $f(\infty) = 0$ for unbounded R. The function μ is the unique solution of the integral equation

$$(\mathbf{I} - \mathbf{N})\mu = -\mathbf{M}\gamma \tag{16}$$

and the piecewise constant function h is given by

$$h = [\mathbf{M}\mu - (\mathbf{I} - \mathbf{N})\gamma]/2. \tag{17}$$

It is clear that the function γ depends on z_0, then so are the functions h and μ. Hence the function $f(z)$ depends also on z_0. One can write $f(z;z_0)$ to emphasize the dependence of the function f on z_0. However, for simplicity of notation, we write $f(z)$ instead of $f(z;z_0)$.

III.1 Bounded R

The function

$$(z-\alpha)f(z) - h_0$$

is analytic in R and its boundary values on Γ satisfies

$$(\eta(t)-\alpha)f(\eta(t)) - h_0 = \gamma(t) + \hat{h}(t) + \mathrm{i}\mu(t), \quad t \in J, \tag{18}$$

where $\hat{h}(t) = (0, h_1 - h_0, \ldots, h_m - h_0)$. Hence, the function

$$-\mathrm{Re}[(z-\alpha)f(z)] + h_0$$

satisfies the assumptions on the function $g(z;z_0)$ in Definition 1, i.e.,

$$g(z;z_0) = -\mathrm{Re}[(z-\alpha)f(z)] + h_0. \tag{19}$$

Then the hydrodynamic Green's function $G(z;z_0)$ is given by

$$G(z;z_0) = \mathrm{Re}[(z-\alpha)f(z)] - h_0 - \frac{1}{2\pi}\log|z - z_0|. \tag{20}$$

It is clear that, the function $G(z;z_0)$ defined by (20) for bounded R satisfies the conditions of Definition 1.

III.2 Unbounded R

The function $f(z)$ is analytic in R with $f(\infty) = 0$ and its boundary values on Γ satisfies

$$f(\eta(t)) = \gamma(t) + h(t) + \mathrm{i}\mu(t), \quad t \in J, \tag{21}$$

where $h(t) = (h_1, \ldots, h_m)$. Thus, the function

$$g(z; z_0) = -Re[f(z)] \tag{22}$$

satisfies the assumptions on the function $g(z; z_0)$ in Definition 1. Hence the hydrodynamic Green's function $G(z; z_0)$ is given by

$$G(z; z_0) = \mathrm{Re}[f(z)] - \frac{1}{2\pi} \log|z - z_0|, \tag{23}$$

which satisfies the conditions of Definition 1.

IV POINT VORTICES

We consider the problem of computing the stream function of fluid motion due to l point vortices in the multiply connected domain R. We have the following theorem from Lin [1] (see also [4]).

Theorem 4 (Lin [1]) *If l vortices of strengths κ_j $(j = 1, 2, \ldots, l)$ are present in an incompressible fluid at the points z_j $(j = 1, 2, \ldots, l)$ in the above domain R, the stream function of fluid motion is given by*

$$\psi(z; z_1, z_2, \ldots, z_l) = \psi_0(z) + \sum_{j=1}^{l} \kappa_j G(z; z_j) \tag{24}$$

where G is the hydrodynamic Green's function and $\psi_0(z)$ is the stream function of the motion due to outside agencies, independent of z_j and κ_j.

In this paper, we shall consider only the stream function of fluid motion due to the l point vortices in the absence of motion due to outside agencies, i.e. we shall assume that $\psi_0(z) = 0$. The stream function of fluid motion is then given by

$$\psi(z; z_1, z_2, \ldots, z_l) = \sum_{j=1}^{l} \kappa_j G(z; z_j). \tag{25}$$

V NUMERICAL IMPLEMENTATION

The boundary integral equation (16) can be solved accurately by the Nyström method with the trapezoidal rule [25, 26] (see [20–23] for more details). For $j = 0$ (for bounded R),1,2,...,m, each interval J_j is discretized by n equidistant nodes. Hence, the total number of nodes in the total parameter domain J is $(m + 1)n$ for bounded R and mn for unbounded R. For domains with piecewise smooth boundaries, singularity subtraction [27] and the trapezoidal rule with a graded mesh [28] are used. By discretizing the integral equation (16) by the Nyström method with the trapezoidal rule, we obtain $(m + 1)n \times (m + 1)n$ linear system for bounded R and $mn \times mn$ linear system for unbounded R. These linear systems are solved by the generalized minimal residual (GMRES) method [29]. Each iteration of the GMRES method requires a matrix-vector product which can be computed using the Fast Multipole Method (FMM) in $O(mn)$ operations [30, 31].

By Theorem 2, the eigenvalues of the operator $\mathbf{N}$ are real in the interval $[-1, 1)$. Since $\mathbf{N}$ is compact, the only possible accumulation point of the eigenvalues is 0 [26, p. 40]. Thus, the eigenvalues of the operator $\mathbf{I} - \mathbf{N}$ are real in the interval $(0, 2]$ with 1 as the only possible accumulation point of these eigenvalues. Hence, for sufficiently large n, the eigenvalues of the matrix of the discretized linear system are real in the interval $(0, 2]$ and clustered around 1 (see [23] for more details). Clustering the eigenvalues of the matrix of the linear system around 1 often results in rapid convergence of the GMRES method. Usually, it will need few iterations for convergence. No preconditioning procedure is required.

In the numerical examples below, the MATLAB function `FBIE` in [22] will be used to obtain approximations to the unique solution μ of the integral equation (16) and the function h in (17), respectively. In the function `FBIE`, the discretized linear system is solved using the MATLAB function `gmres` where the matrix-vector product is computed using the MATLAB function `zfmm2dpart` in the MATLAB toolbox FMMLIB2D developed by Greengard and Gimbutas [30]. In the MATLAB function `FBIE` we choose the parameters `iprec=5` (i.e., the tolerance of the FMM is 0.5×10^{-15}), `restart=10` (i.e., the GMRES method is restarted every 10 inner iterations), `gmrestol`=10^{-14} (i.e., the tolerance of the GMRES method is 10^{-14}), `maxit=10` (i.e., the maximum number of outer iterations of GMRES method is 10). The MATLAB function `FBIE` requires $O(mn \log n)$ operations. Hence, the computational cost for solving the integral equation (16) and computing the function h in (17) is $O(mn \log n)$ operations. For more details, we refer the reader to [22, 23, 30].

By obtaining approximations to μ and h, we thus obtain approximations to the boundary vales of the function f by

$$f = \frac{\gamma + h + \mathrm{i}\mu}{A}.$$

The values of the function f for interior points $z \in R$ can be computed by Cauchy's integral formula. A fast and accurate method to compute the Cauchy integral formula has been given in [22, 23] (see also [32, 33]). The method is based on using the MATLAB function `zfmm2dpart` in [30]. To compute the Cauchy integral formula at p interior points, the method requires $O(p + mn)$ operations [22].

Hence computing the values of the hydrodynamic Green's function at p interior points in R requires $O(mn \log n)$ operations for computing the boundary values of the auxiliary function $f(z)$ and $O(p + mn)$ operations for computing the Cauchy integral formula.

VI NUMERICAL EXAMPLES

We consider three numerical examples to validate the ability of the presented method to handle wide range of multiply connected domains. In the first example, we consider an unbounded multiply connected circular domain R. In the second example, we consider a bounded multiply connected domain R with piecewise smooth boundaries. Finally, we consider in the third examples a real-world unbounded multiply connected domain R with high connectivity and complicated boundaries. In all examples, we consider an even number l of vortices at the points z_j for $j = 1, 2, \ldots, l$. Half of these vortices have strengths $+1$ and the other half have strengths -1. For bounded R, we choose p equidistant points in the domain R. For unbounded R, we choose p equidistant points in a small part of the domain R surrounding the boundary Γ. Then, we compute the values of the hydrodynamic Green's function $G(z; z_j)$ at these p points z for each z_j. The stream function of fluid motion due to the l point vortices in the absence of motion due to outside agencies is then computed from (25). For each example, we plot the streamlines of the stream function and compute the number of GMRES iterations and the total CPU time required for computing the values of $G(z; z_j)$ at the p points z for each z_j.

Example 1 *The domain R is an unbounded multiply connected circular domain of connectivity* 103 *exterior to* 103 *circles.*

For this example, the number of nodes in the discretization of each boundary component is $n = 1024$, the total number of nodes is 105472, the number of point vortices is $l = 20$, and the number of equidistant points chosen in the domain R is $p = 880098$. The GMRES method needs around 22 iterations for converges (see Figure 8 (top)). The streamlines of the stream function corresponding to the 20 point vortices are shown in Figure 3.

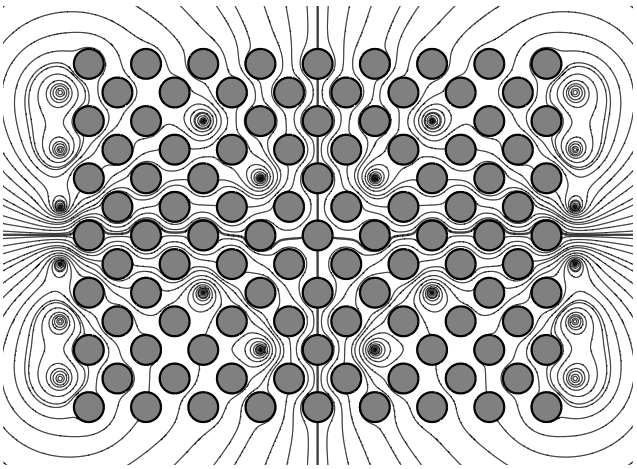

Figure 3. The streamlines for Example 1.

Example 2 *The domain R is a bounded multiply connected domain of connectivity* 5 *interior to a square and exterior to* 4 *other squares.*

For this example, the number of nodes in the discretization of each boundary component is $n = 4096$, the total number of nodes is 20480, the number of point vortices is $l = 8$, and the number of equidistant points chosen in the domain R is $p = 906789$. The GMRES method needs around 26 iterations for converges (see Figure 8 (middle)). The streamlines of the stream function corresponding to the 8 point vortices are shown in Figure 4.

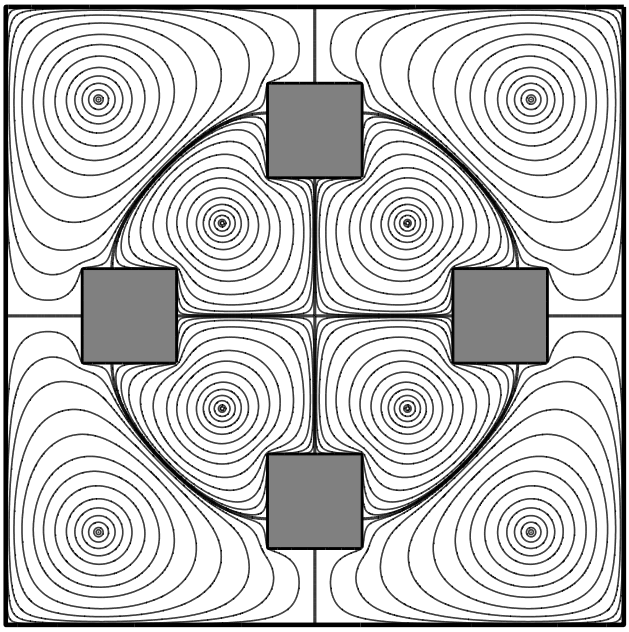

Figure 4. The streamlines for Example 2.

Example 3 *In this example, we consider application of the presented method to a real world problem. We consider the unbounded domain R of connectivity* 210 *exterior to an artificial archipelago located in the waters of the Arabian Gulf,* 4 *kilometres off the coast of Dubai, and known as "The World Islands" (see Fig. 8.15 in [22]).*

An aerial image of "The World Islands" is shown in Figure 5. The boundaries of the islands extracted from the aerial image are shown in Figure 6. The boundaries are parameterized by trigonometric interpolating polynomials. It is clear from Figure 6 that the boundaries are very close to each other, but they do not touch each other. For generalized Neumann kernel formed with the function A given by (6), the presented method gives accurate results even for domains with close-to-touching boundaries (see [22]). However, we need to use large values of nodes n. For this example, we use $n = 8192$ so the total number of nodes is 1720320. The number of point vortices is $l = 16$ and the number of equidistant points chosen in the domain R is $p = 854648$. Since the boundaries are very close to each other and have complex geometry, the GMRES method requires more iterations for convergence. It converges after around 90 iterations which is acceptable for such complicated domains

(see Figure 8 (bottom)). The streamlines of the stream function corresponding to 16 vortices are shown in Figure 7.

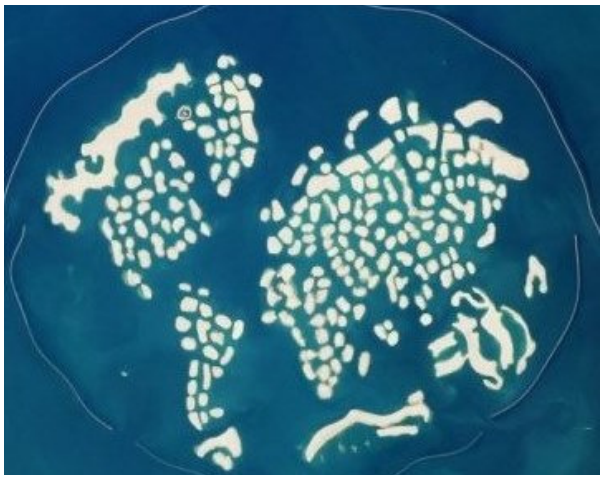

Figure 5. An aerial photograph of "The World Islands" for Example 3.

Figure 6. The boundaries of the islands extracted from the image of "The World Islands" for Example 3.

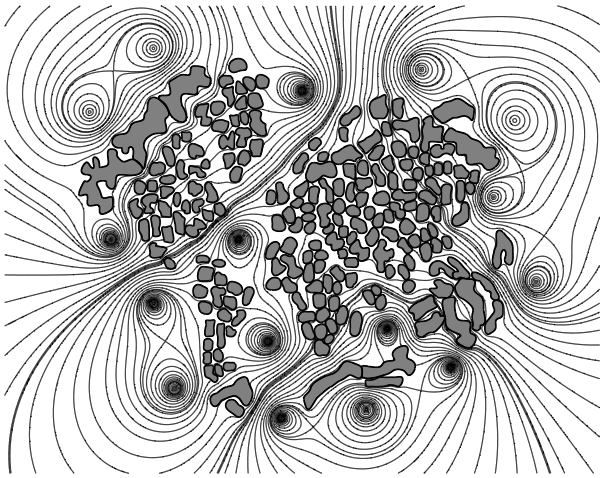

Figure 7. The streamlines for Example 3 obtained with $n = 8192$.

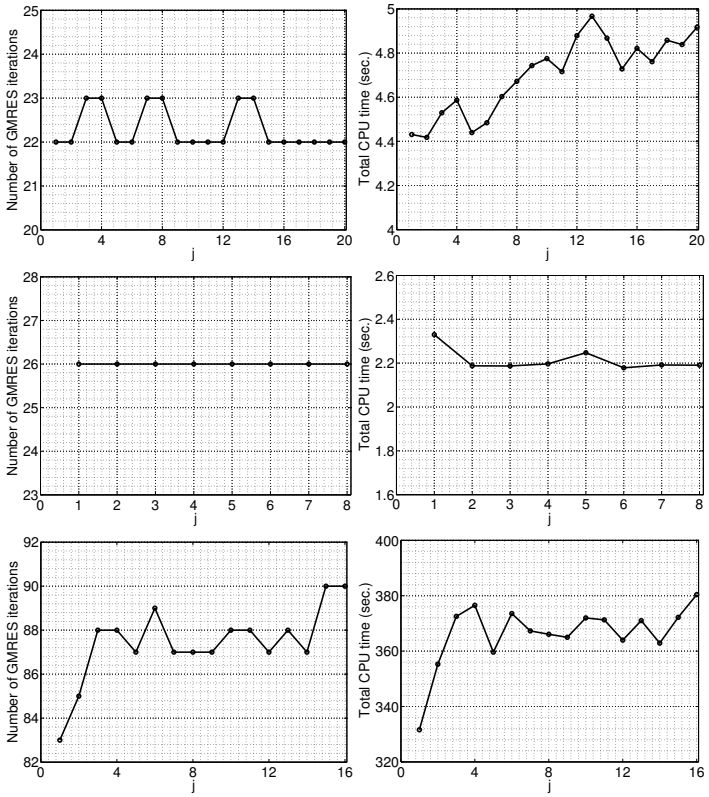

Figure 8. The number of GMRES iterations and the total CPU time required for computing the values of the hydrodynamic Green's function $G(z;z_j)$ at p points z in the domain R for each z_j, $j = 1,2,\ldots,l$, for Example 1 (top), Example 2 (middle), and Example 3 (bottom).

VII CONCLUSIONS

This paper presented a new fast and accurate numerical method for computing the hydrodynamic Green's function in multiply connected domains. The method is based on the boundary integral equations with the generalized Neumann kernel. By solving the integral equation, the hydrodynamic Green's function in the domain R is computed by the Cauchy integral formula. Numerical examples were presented to illustrate that the presented method can be used even for real-world problem domains with complex geometry and high connectivity.

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