THROUGHFLOW AND MAGNETIC FIELD EFFECTS ON THE ONSET OF CONVECTION IN A HELE-SHAW CELL

EFECTOS DEL FLUJO Y EL CAMPO MAGNÉTICO SOBRE EL COMIENZO DE LA CONVECCIÓN EN UNA CELDA DE HELE-SHAW

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An analytical investigation of the combined effects of throughflow and magnetic field on the convective instability in an electrically conducting fluid layer, bounded in a Hele-Shaw cell is presented within the context of linear stability theory. The outcome of the important parameters on the stability of the system is examined analytically as well as graphically. It is observed that the throughflow and magnetic field have both stabilizing effects, while the Hele-Shaw number has a destabilizing effect on the behavior of the system. It is also found that the oscillatory mode of convection is possible only when the magnetic Prandtl number takes the values less than unity.

Se presenta una investigación analítica de los efectos combinados del flujo y el campo magnético en la inestabilidad convectiva de una capa fluida conductora de electricidad, confinada en una celda de Hele-Shaw, en el contexto de la teoría lineal de la estabilidad. Se examinan los resultados para los parámetros relevantes tanto analítica como gráficamente. Se observa que tanto el flujo como el campo magnético, tienen efectos estabilizantes, mientras que el número de Hele-Shaw tiene un efecto desestabilizante. También se encuentra que el modo de convección oscilatorio es posible sólo cuando el número magnético de Prandtl es menor que la unidad.

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I. INTRODUCTION

This paper deals with the joint effect of magnetic field and throughflow on the convective instability of an electrically conducting fluid layer confined vertically by two thermally insulated planes and horizontally by two perfect heat conducting planes. This configuration is called the Hele-Shaw cell. A range of applications of fluid mechanics goes to the Hele-Shaw flows. Nowadays, this flow is applied in numerous fields of physics and engineering, in particular, material processing and crystal growth owing to manufacturing procedures [1–3]. Also, during the last years, convection in a porous medium has gained significant interest because of its importance in geothermal, petroleum production, oil reservoir, building of thermal insulations and nuclear reactor [4]. The problem in a porous medium can be easily solved by taking an appropriate permeability of the Hele-Shaw cell. Hele–Shaw [5] was the first who pointed out the similarity between the two-dimensional flow in a porous medium and Hele-Shaw cell by taking an equivalent permeability for the Hele-Shaw cell, where is the width of the cell. He showed that the Hele-Shaw cell can be a controlling device for quantitative study of two-dimensional flow in porous media by suitably recognizing the Hele-Shaw permeability. The similarity among flow in a porous medium and flow in a Hele-Shaw cell has commonly been applied to study convection in the former [6–8].

The convective instability of electrically conducting fluid in the existence of a magnetic field has drawn great attention as a consequence of its many real-world applications for instance in electrical machineries, chemical apparatus, plasmas, MHD accelerators and power generation systems. The study of magnetic field on the onset of convection yields a range of activities when the ratio of the magnetic to thermal diffusivity is small; the governing system then allows both stationary and oscillatory mode of convections. The strength of magneto convection is indicated by the Chandrasekhar number, which is the relation between the Lorentz force and the viscosity. If the Lorentz force was smaller than the viscous force, then the convective motions twist and stretch the magnetic field. If the Lorentz force was larger than the viscous force, then the magnetic field sets the plasma flows along the field direction and constrains the convection. The large numbers of investigations related to magneto convection are recognized by Chandrasekhar [9] and Nield and Bejan [4]. Thompson [10] and Chandrasekhar [9] were the first to examine the effect of magnetic fields on the convective instability. Rudraiah and Shivakumara [11] studied convection with an imposed magnetic field. They observed that the magnetic field, under some situations, makes the system unstable. The interplay between magnetic fields and convection by considering the effect of a solid rotor on a non-uniform magnetic field was investigated by Weiss [12]. Abd-el-Malek and Helal [13] studied the problem of an unsteady convective laminar flow under the effect of a magnetic field. They found that the velocity boundary-layer thickness becomes smaller for the increase in the magnetic influence number. Very recently, the effect of magnetic field on the convective instability in nanofluids was considered by Yadav et al. [14–18], Chand and Rana [19,20], Sheikholeslami et al. [21–23], Al-Zamily [24], Gupta et al. [25] and Hamada et al. [26].

The throughflow effect on the convective instability in an electrically conducting fluid layer with magnetic field is an important concept because of its applications in engineering, geophysics and magneto-hydrodynamics. In situ processing of electronic components, chemical equipment, cooling of nuclear reactors, energy assets such as coal, geothermal energy, oil shale and many real-world problems frequently occupy the throughflow in a Hele-Shaw cell. The significance of buoyancy-driven convective instability in such circumstances may become important when specific processing is needed. Besides, the throughflow effect in such situations offers the opportunity of controlling the convective instability by regulating the throughflow in accumulation to the gravity. Throughflow changes the basic temperature profile from linear to nonlinear with layer height, which influences the stability of the system considerably. The effect of throughflow on the onset of convection was studied by Jones and Persichetti [27]. Its extension to porous medium was made by Wooding [28], Sutton [29] and Nield [30]. They observed that the effect of throughflow is not always stabilizing and depends on the character of the boundaries. Khalili and Shivakumara [31] examined the effect of throughflow on the onset of convection in a porous medium with internal heat generation. They observed that throughflow destabilizes the system in the presence of an internal heat source, even if the boundaries are of the same type. Later on, many investigators studied the effect of throughflow on convective instability for different types of fluids [32–36].

However, no study has been found in the literature which considers the effect of throughflow on magneto convection confined within a Hele-Shaw cell. Therefore, here we examine the combined effect of throughflow and magnetic field on the convective instability in an electrically conducting fluid layer, bounded within a Hele-Shaw cell. By linear stability theory, the critical conditions for stationary and oscillatory convections are derived analytically, and discussed graphically.

II. MATHEMATICAL MODEL

In this work, an infinitely extended horizontal incompressible electrically conducting fluid layer of height d is considered. The fluid layer is confined between two parallel boundaries at $z^* = 0$ and $z^* = d$ which are preserved at uniform but different temperatures T_l^* and T_u^* ($T_l^* > T_u^*$), respectively. The fluid shall be infinite in the x−direction, but restricted in the y−direction by sidewalls at $y^* = 0$ and $y^* = b$. For a suitably small thickness, $b \ll d$, the flow can be estimated as a 2−dimensional Stokes flow in the $x - z$−plane, usually called a Hele-Shaw flow. A constant magnetic field $H^* = (0, 0, H_0^*)$ is applied. The physical configuration of the system is shown in figure 1. Asterisks are used to differentiate the dimensional variables from the non-dimensional variables (without asterisks).

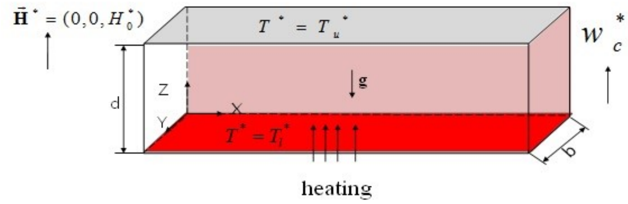

Figure 1. The physical configuration of the system.

By considering the Hele-Shaw approximation and using the Boussinesq approximation, the governing equations under this model are:

$$\nabla^* \vec{v}^* = 0, \tag{1}$$

$$\frac{\mu}{K}\vec{v}^* = -\nabla^* p^* + \rho_0 \left[1 - \beta(T^* - T_u^*)\right] \vec{g} + \mu_e (\vec{H}^* \cdot \nabla^*)\vec{H}^*, \tag{2}$$

$$\left[\frac{\partial}{\partial t^*} + (\vec{v}^* \cdot \nabla^*)\right] T^* = \alpha \nabla^{*^2} T^*, \tag{3}$$

$$\left[\frac{\partial}{\partial t^*} + (\vec{v}^* \cdot \nabla^*)\right] \vec{H}^* = \nu_m \nabla^{*^2} \vec{H}^* + (\vec{H}^* \cdot \nabla^*)\vec{v}^*, \tag{4}$$

$$\nabla^* \vec{H}^* = 0. \tag{5}$$

Here, $\vec{v}^*$ is the velocity of the fluid, t^* is the time, ρ_0 is the fluid density at the reference temperature T_u^*, p^* is the pressure, $\vec{H}^*$ is the magnetic field, β is the thermal expansion coefficient, α is the thermal diffusivity, $K = b^2/12$ is the permeability of the fluid flow in the Hele Shaw cell, μ, ν_m, μ_e and k are the viscosity, magnetic viscosity, magnetic permeability and thermal conductivity of the fluid, respectively. As stated before, eqs. 1,2,3,4,5 are written under Boussinesq approximation, which neglects density differences except where they appear in terms multiplied by gavity's acceleration.

We assume that there is an upward throughflow with constant mean value w_u^*. Thus the boundary situations are:

$$w^* = w_c^*, \quad T^* = T_l^*, \quad \text{at} \quad z^* = 0, \tag{6a}$$
$$w^* = w_c^*, \quad T^* = T_u^*, \quad \text{at} \quad z^* = d. \tag{6b}$$

We define the following non dimensional variables:

$$x = \frac{x^*}{d}, t = \frac{t^*}{d^2}\alpha, p = \frac{p^* d^2}{\mu\alpha}, T = \frac{T^* - T_u^*}{T_l^* - T_u^*}, \vec{v} = \frac{d}{\alpha}\vec{v}^*, \vec{H} = \frac{\vec{H}^*}{H_0^*}. \tag{7}$$

The governing equations then become:

$$\nabla \vec{v} = 0, \tag{8}$$

$$\frac{\vec{v}}{H_s} = -\nabla^p + R_a T\hat{e}_z + QP_m(\vec{H}\cdot\nabla)\vec{H}, \tag{9}$$

$$\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)\right]T = \nabla^2 T, \tag{10}$$

$$\left[\frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)\right]\vec{H} = (\vec{H}\cdot\nabla)\vec{v} + P_m\nabla^2\vec{H}, \tag{11}$$

$$\nabla\vec{H} = 0. \tag{12}$$

In the non-dimensional form, the boundary conditions become:

$$w = \lambda,\;\; T = 1,\;\; \text{at}\;\; z = 0, \tag{13a}$$
$$w = \lambda,\;\; T = 0^*,\;\; \text{at}\;\; z = 0. \tag{13b}$$

The non-dimensional parameters in eqs. 8-12 are $R_a = gd^3\beta\Delta T/\alpha\nu$ (Rayleigh number), $H_s = K/d^2$ (Hele-Shaw number), $P_m = \nu_m/\alpha$ (magnetic Prandtl number), $Q = \mu_e H_0^{*2} d^2/\rho_0\nu\nu_m$ (Chandrasekhar number), $\nu = \mu/\rho_0$ (kinematic viscosity), $\nu_m = \mu_e/\rho_0$ (magnetic viscosity) and $\lambda = dw_c^*/\alpha$ (Pèclet number).

II.1. Basic State

The basic state of the fluid is considered time-independent, and is given by

$$\vec{v}_b = \lambda\hat{e}_z,\;\; T = T_b,\;\; p = p_b,\;\; \vec{H}_b = \hat{e}_z. \tag{14}$$

Then eq. 10 gives:

$$\frac{d^2T_b}{dz^2} - \lambda\frac{dT_b}{dz} = 0. \tag{15}$$

The boundary conditions for $T_b(z)$ are:

$$T_b = 1\;\; \text{at}\;\; z = 0,\;\; T_b = 0\;\; \text{at}\;\; z = 1. \tag{16}$$

With the application of the boundary conditions 16, the solution of eq. 15 is

$$T_b(z) = \frac{e^\lambda - e^{\lambda z}}{e^\lambda - 1} \tag{17}$$

II.2. Perturbation theory

We now apply small perturbations on this basic state as:

$$\vec{v} = \vec{v}_b + \vec{v}\,',\;\; T = T_b + T',\;\; p = p_b + p',\;\; \vec{H} = \hat{e}_z + \vec{H}\,' \tag{18}$$

where the primed quantities are functions of x and t.

After substituting the eq. 18 into eqs. 8-**??**, and linearizing the equations, we get:

$$\nabla\vec{v}\,' = 0, \tag{19}$$

$$\frac{\vec{v}\,'}{H_s} = -\nabla p' + R_a T'\hat{e}_z + QP_m\frac{\partial\vec{H}\,'}{\partial z}, \tag{20}$$

$$\frac{\partial T'}{\partial t} + \vec{v}\,'\cdot\nabla T_b + \vec{v}_b\cdot\nabla T' = \nabla^2 T', \tag{21}$$

$$\frac{\vec{H}\,'}{\partial t} + \vec{v}\,'\cdot\vec{H}_b\,' + \vec{v}_b\,'\cdot\vec{H}\,' = \vec{H}\,'\cdot\nabla\vec{v}_b + \vec{H}_b\cdot\nabla\vec{v}\,' + P_m\nabla^2\vec{H}\,' \tag{22}$$

$$\nabla\vec{H}\,' = 0. \tag{23}$$

Operating on eq. 21 with $\hat{e}_z\cdot\nabla$ and using the eqs. (19) and (23), we obtain the -component of the momentum equation as:

$$\frac{\nabla^2 w'}{H_s} - R_a\nabla_p^2 T' - QP_m\nabla^2\left[\frac{\partial H_z'}{\partial z}\right] = 0. \tag{24}$$

The z-component of the eq. 22 is

$$\frac{\partial H_z'}{\partial t} + \lambda\frac{\partial H_z'}{\partial z} = \frac{\partial w'}{\partial z} + P_m\nabla^2 H_z', \tag{25}$$

eliminating from eqs. 24 and 25, we get:

$$\left(P_m\nabla^2 - \frac{\partial}{\partial t} - \lambda\frac{\partial}{\partial z}\right)\left[\frac{\nabla^2 w'}{H_s} - R_a\nabla_p^2 T'\right] + QP_m\nabla\left[\frac{\partial w'}{\partial z}\right] = 0, \tag{26}$$

taking the perturbation quantities in the form:

$$(w', T') = [W(z), \Theta(z)]\exp[ik_x x + ik_y y + st], \tag{27}$$

where k_x and k_y are the wave numbers in the x and y directions, respectively and s is the growth rate of volatility. The growth rate s is commonly a complex number such that $s = s_r + is_i$. The case $s_r < 0$ means all time stability, while the system is unstable when $s_r > 0$. For neutral stability, the real part of s is zero. Hence, we consider $s = is_i$, where s_i is real and is a dimensionless frequency.

After inserting eq. 27 into eqs. 26 and 21, we have:

$$\begin{aligned}[P_m(D^2 - a^2) - is_i - \lambda D]\left[(D^2 - a^2)\frac{W}{H_s} + a^2 R_a\Theta\right]&\\ + QP_m(D^2 - a^2)D^2 W = 0,&\end{aligned} \tag{28}$$

$$(D^2 - a^2 - \lambda D - is_i)\Theta - fW = 0, \tag{29}$$

where $d/dz \equiv D$, $f(z) = \lambda e^{\lambda z}/(1 - e^\lambda)$ and $a = \sqrt{k_x^2 + k_y^2}$ is the resulting dimensionless wave number.

In the perturbation dimensionless form, the boundary conditions become:

$$W = 0,\;\; \Theta = 0,\;\; \text{at}\;\; z = 0, 1 \tag{30}$$

In order to solve the system of equations 28-30, the Galerkin weighted residuals method is applied. Accordingly, the support functions for and are assumed as:

$$W = \sum_{P=1}^{N} A_P W_P, \quad \Theta = \sum_{P=1}^{N} B_P \Theta_P, \tag{31}$$

where $W_P = \Theta_P = \sin p\pi z$ (fulfilling the boundary conditions), and A_P and B_P are unknown coefficients, and $p = 1, 2, 3, ..., N$. On putting the above expression for W and Θ into eqs. 28-29, we get a system of $2N$ linear algebraic equations in the $2N$ unknowns A_P and B_P where $p = 1, 2, 3, ..., N$. For the occurrence of non-trivial solutions, the determinant of coefficients matrix must be zero, which provides the characteristic equation for the system with Rayleigh number R_a as the eigenvalue.

III. RESULTS AND DISCUSSION

To get analytical results, we choose so the Darcy-Rayleigh number R_a is given by

$$R_a = \Delta_1 + i s_1 \Delta_2, \tag{32}$$

where

$$\Delta_1 = \frac{J(\lambda^2 + 4\pi^2)\left[J(s_i^2 + J^2 P_m^2) + H_s \pi^2 P_m (s_i^2 + J^2 P_m) Q\right]}{4a^2 H_s \pi^2 (s_i^2 + J^2 P_m^2)} \tag{33}$$

$$\Delta_2 = \frac{J(\lambda^2 + 4\pi^2)\left[s_i^2 + J P_m (J P_m + H_s \pi^2 (P_m - 1) Q)\right]}{4a^2 H_s \pi^2 (s_i^2 + J^2 P_m^2)} \tag{34}$$

Here, $J = (a^2 + pi^2)$.

Since R_a is a physical quantity, it must be real. Thus, it follows from eq. 33 that either $s_i = 0$ (stationary convection) or $\Delta_2 = 0$ ($s_i \neq 0$ non-oscillatory convection).

III.1. *Stationary mode of convection*

Stationary convection occurs when $s_i = 0$. In this case, from eq. 33, the stationary Rayleigh number R_a^S can be obtained as

$$R_a^S = \frac{(a^2 + \pi^2)(\lambda^2 + 4\pi^2)\left[(a^2 + \pi^2) + H_s \pi^2 Q\right]}{4a^2 H_s \pi^2} \tag{35}$$

From the eq. 35, it is clear that the critical Rayleigh number increases with an increase in Q and λ, while decreases with H_s. Thus, the magnetic field and the throughflow have a stabilizing effect, while the Hele Shaw number has a destabilizing effect on the system.

The critical wave number a_c can be obtained as

$$a_c = \pi(1 + H_s Q)^{1/4}. \tag{36}$$

For the case of porous medium ($H_s = 1$), eqs. 35 and 36 become:

$$R_a^S = \frac{(a^2 + \pi^2)(\lambda^2 + 4\pi^2)\left[(a^2 + \pi^2) + \pi^2 Q\right]}{4a^2 \pi^2}, \tag{37}$$

$$a_c = \pi(1 + Q)^{1/4}. \tag{38}$$

If there is no magnetic field ($Q = 0$), eqs. 37 and 38 become:

$$R_a^S = \frac{(a^2 + \pi^2)^2}{a^2}\left[1 + \frac{\lambda^2}{4\pi^2}\right], \tag{39}$$

$$a_c = \pi. \tag{40}$$

This result is identical to that found by Nield and Kuznetsov [37].

From eqs. 39 and 40, when the throughflow is equal to one, i.e. $\lambda = 1$, the critical Rayleigh number is 40.4784. Recently, Barletta et al. [38] obtained a more exact value by using a different methodology, getting 40.8751. Hence the approximation formula used in this paper gives an accuracy of 1 %. This shows that the approximation used in this paper is satisfactory for the case when throughflow is equal to one.

In the absence of throughflow, i.e. $\lambda = 0$ eq. 37 gives

$$R_a^S = \frac{(a^2 + \pi^2)\left[(a^2 + \pi^2) + \pi^2 Q\right]}{a^2}. \tag{41}$$

Eq. 41 coincides with that of Kiran et al. [39].

III.2. *Oscillatory mode of convection*

For oscillatory convection $\Delta_2 = 0$ and $s_i \neq 0$. Using these in eq. 33, the expressions for oscillatory Rayleigh number R_a^{Osc} and the frequency of oscillations s_i can be written as:

$$R_a^{Osc} = \frac{J(\lambda^2 + 4\pi^2)\left[J(s_i^2 + J^2 P_m^2) + H_s \pi^2 P_m (s_i^2 + J^2 P_m^2) Q\right]}{4a^2 H_s \pi^2 (s_i^2 + J^2 P_m^2)}. \tag{42}$$

$$s_i = -J P_m \left[J P_m + H_s \pi^2 (P_m - 1) Q\right] \tag{43}$$

Eq. 43 shows that the necessary condition for the occurrence of oscillatory mode of convection is:

$$\frac{J P_m}{H_s \pi^2 (1 - P_m)} < Q \tag{44}$$

In order to build Q positive, the magnetic Prandtl number P_m must be less than unity. From eq. 43, it is also found that the oscillatory mode of convection is not likely in the absence of magnetic field.

The graphical representation of the stability of the system in (R_a, a) plane is made in Figs. 2-5 for various parameter values. The values used in the figures are taken from various sources [4, 9, 40–42]. The linear stability theory gives the condition of stability in terms of the critical Rayleigh number, below which the system is stable, and unstable above.

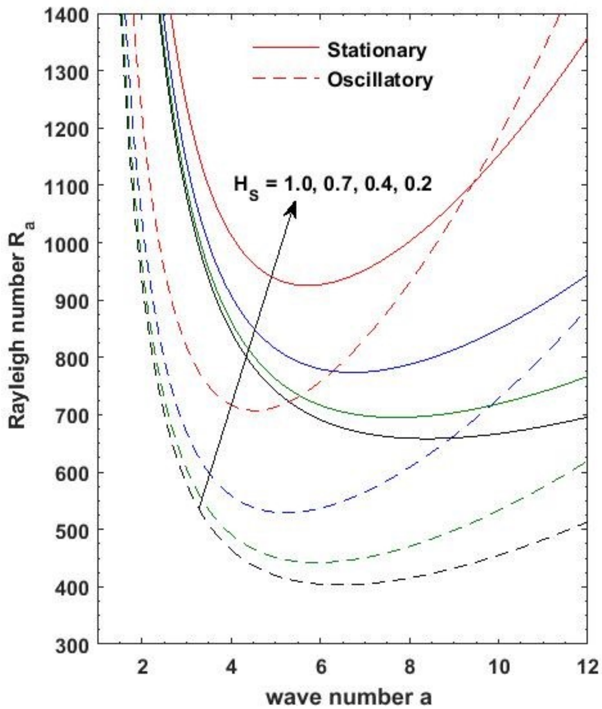

Figure 2. The effect of the Hele-Shaw number H_s on the stationary and oscillatory convection curves at $\lambda = 0.5$, $Q = 50$ and $P_m = 0.5$.

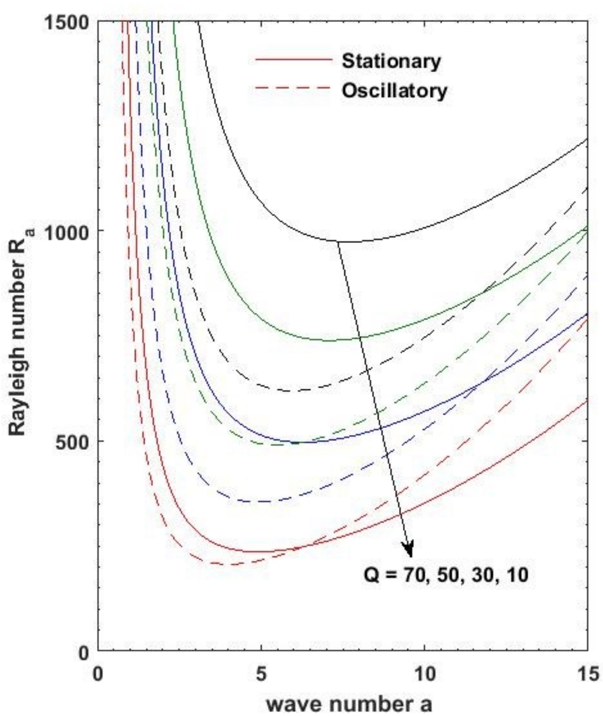

Figure 3. The effect of the magnetic field Q on the stationary and oscillatory convection curves at $\lambda = 0.5$, $H_s = 0.5$ and $P_m = 0.5$.

Figure 2 represents the effect of the Hele-Shaw number H_s on the stability of the system. From the figure it is observed that the critical Rayleigh number increases with a decrease in the Hele-Shaw number H_s. Hence, the Hele-Shaw number has a destabilizing effect on the behavior of the system. This is because on increasing the value of Hele-Shaw number the permeability of the Hele-Shaw cell increases and consequently the width of Hele-Shaw cell increases, which in turn makes the fluid flow faster.

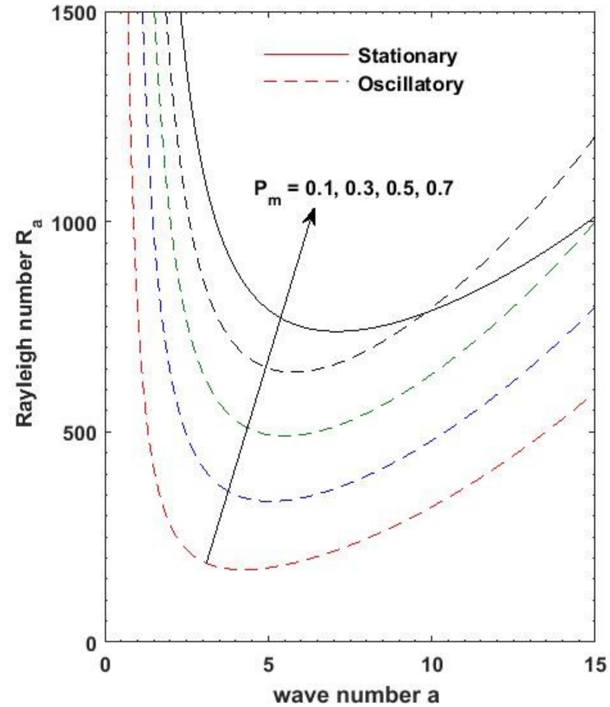

Figure 4. The effect of the through flow λ on the stationary and oscillatory convection curves at $Q = 50$, $H_s = 0.5$ and $P_m = 0.5$.

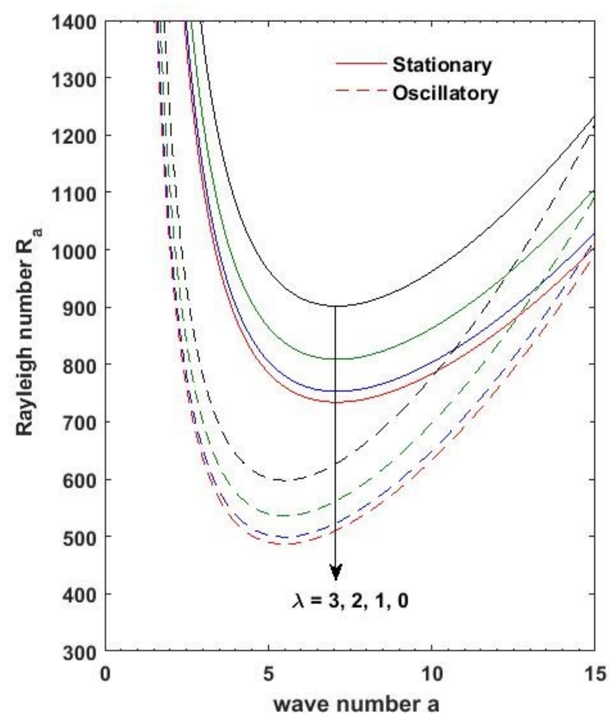

Figure 5. The effect of the magnetic Prandtl number P_m on the stationary and oscillatory convection curves at $Q = 50$, $H_s = 0.5$ and $\cdot\lambda = 0.5$.

The effect of the magnetic field parameter Q on the onset of stationary and oscillatory convection curves are displayed in Figure 3. This figure shows that a decrease in the value of Q decreases the critical stationary and oscillatory Rayleigh numbers. Hence the magnetic field parameter Q delays the onset of convection. This is because the increase of Q increases the Lorentz force, and the Lorentz force gives more resistance to transport. Hence, the magnetic field has a stabilizing effect on the system.

The effect of the throughflow parameter λ on the stationary and oscillatory mode of convections is shown in Figure 4. The minima on each plot give the critical Rayleigh number for the exchange of stabilities. This critical Rayleigh number decreases with decreasing value of the throughflow λ and hence their effect is to delay the onset of convection.

To measure the effect of the magnetic Prandtl number P_m on the stability of the system, the deviation of Rayleigh number for stationary and oscillatory mode of convection is plotted in Figure 5 as a function of wave number a for different values of the magnetic Prandtl number P_m. From this figure it is observed that the magnetic Prandtl number P_m has no effect on the stationary convection, while for oscillatory convection it has a stabilizing effect on the system.

IV. CONCLUSIONS

In this paper, the combined effect of throughflow and magnetic field on the instability of a fluid confined within a Hele-Shaw cell heated from below was investigated using linear stability theory. The behaviour of the magnetic field, the throughflow and the Hele-Shaw number on the onset of convection was analyzed analytically and discussed graphically. The results show that the increase in magnetic field and strength of throwflow tends to stabilize the system. An increase in the Hele-Shaw number was found to have a destabilizing effect on the system. It was also observed that the oscillatory mode of convection is possible only when the magnetic Prandtl number is smaller than unity.

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