THE PARAXIAL APPROXIMATION, REVISITED
LA APROXIMACIÓN PARAXIAL, REVISITADA

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We define the paraxial ray as the ray making infinitesimal angles with a central axis and normal to a reflecting or refracting surface. We show that for some physical situations these conditions can be relaxed, while for others non-compliance with one of them can lead to catastrophic deviations in the behavior of the output light beam. We first derive and analyze the generalized spherical mirror formula. Next, we consider the case of refracting ball and show that there can occur the situation, where the parallel input light beam is split into converging and diverging output beams in it. The issues outlined in this article will be useful for students, when learning the basics of geometrical optics.

Definimos el rayo paraxial como el rayo que forma ángulos infinitesimales con un eje central y normal a una superficie reflectante o refractante. Mostramos que para algunas situaciones físicas estas condiciones se pueden relajar, mientras que para otras el incumplimiento de una de ellas puede conducir a desviaciones catastróficas en el comportamiento del haz de luz de salida. Primero derivamos y analizamos la fórmula del espejo esférico generalizado. A continuación, consideramos el caso de la bola refractora y mostramos que puede ocurrir la situación en la que el haz de luz de entrada paralelo se divide en haces de salida convergentes y divergentes en él. Los temas descritos en este artículo serán útiles para los estudiantes que estudian los conceptos básicos de la óptica geométrica.

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I. INTRODUCTION

The paraxial rays approximation is the basic concept of Gaussian (paraxial) optics by means of which the simple equations for spherical mirror and thin lens can be derived. The discussion of the paraxial approximation is presented in Refs. [1] and [2]. Serway and Jewett [3] talk about the paraxial rays as the rays making small angles with the principal axis. Halliday, Resnick and Walker [4] define the paraxial rays as the rays close to the central axis. In Ref. [5] the rays almost parallel with the central axis and close to it are called paraxial. Thus, the textbook definitions of these rays are rather unclear and imply that one or two conditions should be met. In our opinion, the most precise definition of this concept should be as follows: the paraxial ray is the ray making infinitesimal angles with a central axis and normal to a reflecting or refracting surface. In this paper we show that for some physical situations these conditions can be relaxed, while for others non-compliance with one of them can lead to catastrophic deviations in the behavior of the output light beam. The issues outlined in this article will be useful for students, who study the basics of geometrical optics.

II. THE GENERALIZED SPHERICAL MIRROR FORMULA

Let us consider a spherical mirror with radius of curvature R. In Fig. 1 we set the position of the incident light ray using two parameters: object distance s, measured from vertex V to the point of ray intersection with optic axis and angle φ between this axis and the ray (instead of s we could choose, as a parameter, angle θ between the incident ray and normal to a reflecting surface).

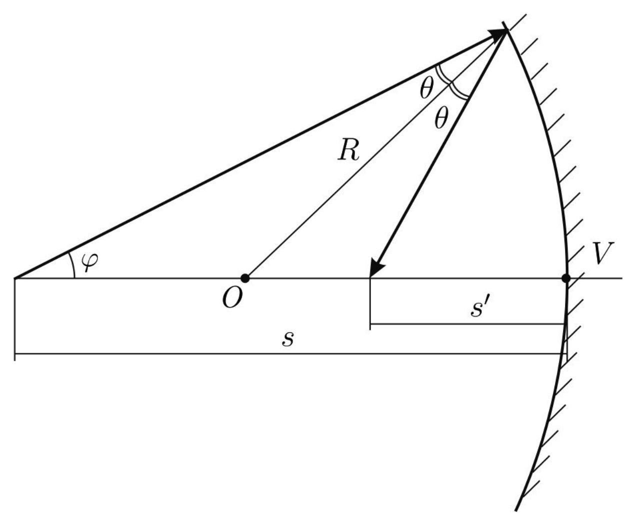

Figure 1. Reflection of an arbitrary ray from a concave mirror.

The image distance s' is measured from vertex V to the point of reflected ray intersection with the optic axis.

According to Cartesian sign convention [6]:

1. Light initially propagates from left to right.
2. Distances measured normal to the optic axis are positive above and negative below.
3. Distances measured in the direction opposite to the direction of incident light are taken as negative. Distances measured in the same direction as the incident light are considered positive.

4. Angles are positive when produced by clockwise rotation from the optic axis or the surface normal, and negative when produced by counter-clockwise rotation.

5. If the vertex lies to the left of the center of curvature, the radius of curvature is positive. If the vertex lies to the right of the center of curvature, the radius of curvature is negative.

Applying the sign convention for the case shown in Fig. 1, we have: $s < 0$, $s' < 0$, $R < 0$, $\varphi < 0$ $(-\pi/2 < \varphi < 0)$, $\theta > 0$. Then, using the law of sines, we get:

$$\frac{R}{\sin\varphi} = \frac{R-s}{\sin\theta}, \tag{1}$$

$$\frac{R}{\sin(2\theta-\varphi)} = \frac{R-s'}{\sin\theta}. \tag{2}$$

Solving this system with respect to s', we find:

$$s' = R\left[1 + \frac{x-1}{2(x-1)^2\sin^2\varphi - 1 - 2(x-1)\cos\varphi\sqrt{1-(x-1)^2\sin^2\varphi}}\right], \tag{3}$$

where $x = s/R$. Therefore, in reality, the value of distance s' depends not only on R and s, but also on angle φ, that is, the spherical mirror always present spherical aberration [1], [7]. Equation (3) is invariant under transformation $\varphi \to -\varphi$, which is consistent with the symmetry of the problem. Despite the fact that this equation is obtained for the case shown in Fig. 1, it is valid for the whole range of object distance values $(-\infty < x < \infty)$.

In the paraxial approximation angles φ and θ are small. Then $\sin\varphi \approx \varphi$, $\sin\theta \approx \theta$, $\sin(2\theta-\varphi) \approx 2\theta - \varphi$ and from equations (1), (2) we derive the well known approximate object-image relationship for spherical mirror:

$$\frac{1}{s} + \frac{1}{s'} \approx \frac{2}{R}. \tag{4}$$

Using equation (4) we obtain the value for the image distance s'_0 in the paraxial approximation:

$$s'_0 = \frac{x}{2x-1}R. \tag{5}$$

Equations (3), (5) allow one to highlight the domain on plane (x, φ), for which equation (5) gives approximately correct value for image distance. In Fig. 2 we present the results of numerical calculation of contour line, along which the relative error $\delta_{s'} = |(s' - s'_0)/s'|$ in the determination of image distance is equal to 5 %.

The domain corresponding to a large value of this quantity is shaded gray.

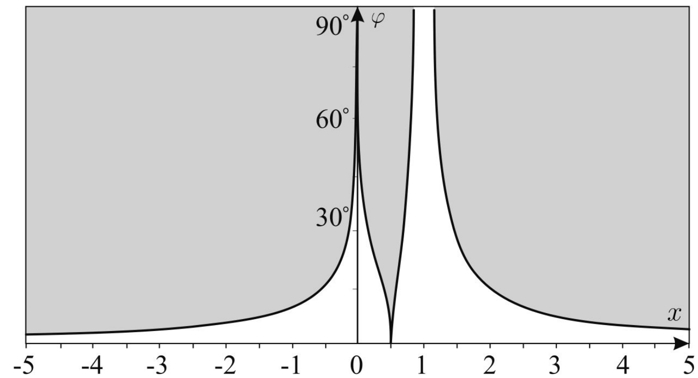

Figure 2. Domains of credibility (white) and incredibility (gray) of the paraxial approximation for a spherical mirror at $\delta_{s'} = 5\,\%$.

It is seen that for large relative object distance ($|x| \gg 1$) both paraxial conditions should be satisfied, that is both φ and θ should be small. For object lying near the mirror's center of curvature the angles of incidence θ are always very small, whereas φ can take arbitrary values. Due to this fact $s \approx s' \approx R$ and equations (4), (5) are well performed (the relative error $\delta_{s'}$ is small). For object lying near the principal focus of mirror ($x = 1/2$) both φ and θ should be again very small, since only within these conditions the reflected ray is almost parallel to the principal axis. At last, for object position close to vertex V ($s \to 0$), both φ and θ can take arbitrary equal values (we remind that in general case these values are related through Eq. (1)). This fact can be explained in the following way. Since $s \to 0$, then from equation (1) we have approximately: $\varphi \approx \theta$. At this rate using equation (2) we get: $s' \to 0$ so that $s \approx -s'$. This equality agrees well with approximate equation (5).

III. THE REFRACTING BALL

It is known that the violation of the second paraxial condition (the ray makes large angles with a normal to a reflecting or refracting surface) leads to the spherical aberration effect [1], [7]. Let us consider the parallel light beam, which is incident on the transparent ball of radius R and relative refractive index $n > 1$. We find distance $d_{\rm cr}$ from the central axis to the incident ray, which, after refraction, falls to the point of intersection of this axis with the ball (Fig. 3).

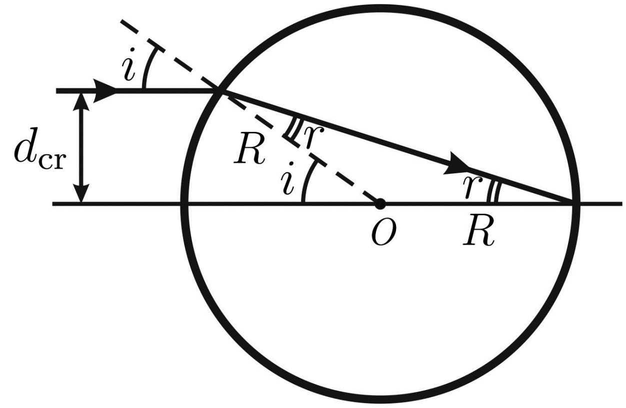

Figure 3. Geometry used to derive equation (9).

Considering Fig. 3 and applying the Snell's law, we get:

$$\sin r = (\sin i)/n. \quad (6)$$

Furthermore

$$r = \frac{i}{2}, \quad (7)$$

$$\sin i = d_{\mathrm{cr}}/R. \quad (8)$$

Solving system of equations (6)-(8) with respect to d_{cr}, we obtain:

$$d_{\mathrm{cr}} = \frac{n\sqrt{4-n^2}}{2}R. \quad (9)$$

In Fig. 4 we plot d_{cr} as the function of n using equation (9).

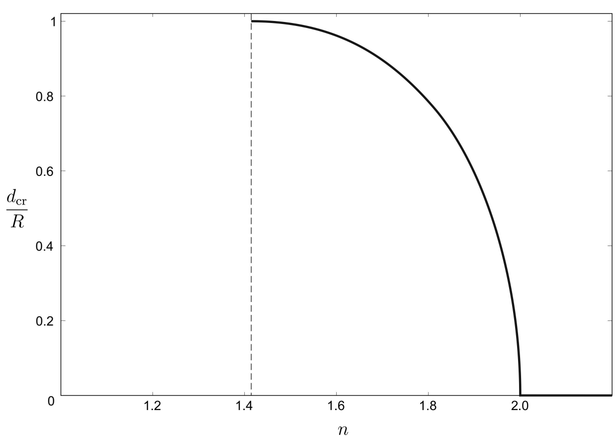

Figure 4. d_{cr} as the function of n according to equation (9).

For $1 < n < \sqrt{2}$ all incident rays converge at various focal points behind the ball (Fig. 5), that is, this ball is a classical converging lens. Another words, the value of $d_{\mathrm{cr}} > R$, so it is impossible to find a ray that, after refraction, falls to the point of intersection of the axis with the refracting sphere.

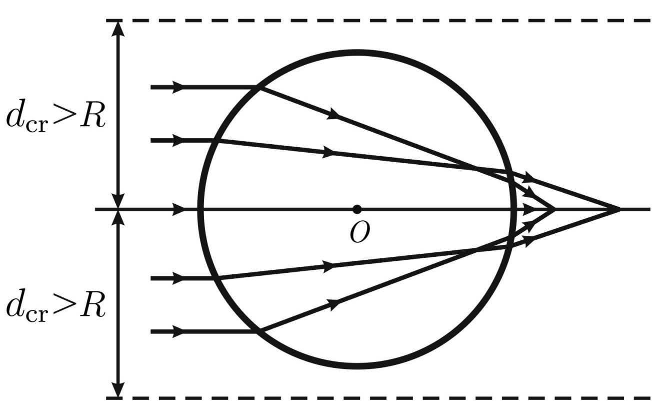

Figure 5. The ray tracing in a refracting ball for $1 < n < \sqrt{2}$.

If n increases, then the refractive power of the ball increases too and for $\sqrt{2} < n < 2$ incident rays lying inside a cylinder of radius $d_{\mathrm{cr}} < R$ are also collected at various focal points behind the ball, whereas peripheral rays with $d_{\mathrm{cr}} < d < R$ intersect at different points lying inside the ball (Fig. 6).

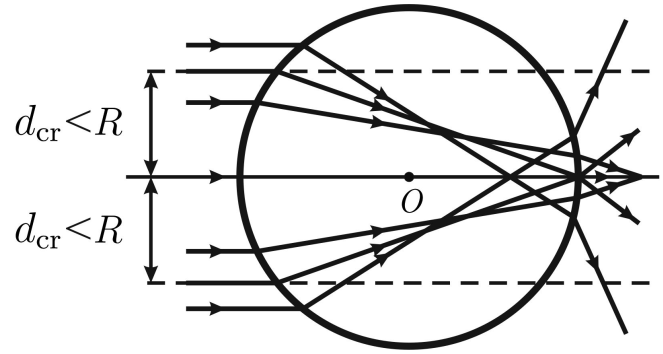

Figure 6. The ray tracing in a refracting ball for $\sqrt{2} < n < 2$.

Therefore, the parallel input light beam is split into converging and diverging output beams. The considered effect is a consequence of the simultaneous violation of the second paraxial condition and the finite thickness of the lens [8] (the refracting ball). Finally, for $n > 2$ this ball is a diverging lens even for the paraxial rays (Fig. 7).

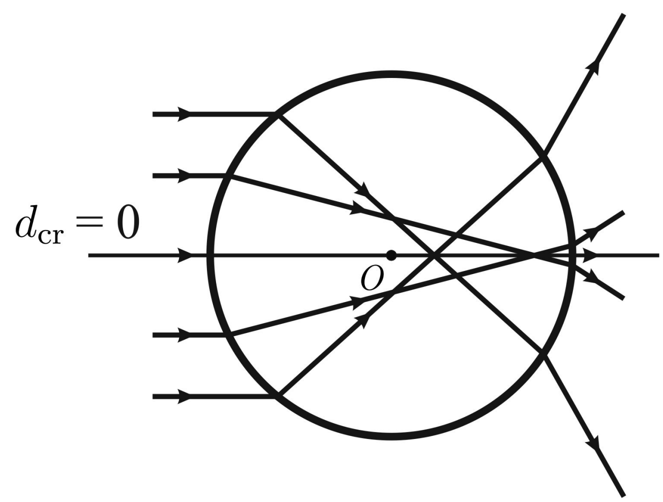

Figure 7. The ray tracing in a refracting ball for $n > 2$.

We should note that at present time there are several ways of creation of the paraxial singlet lenses free of spherical aberrations. For example [9], the selection of the shape of the input surface is carried out so, that the rays inside the lens do not cross each other as well.

IV. CONCLUSION

In this paper we present two useful in geometrical optics teaching examples that facilitate the students grasping such an important concept as paraxial ray. First, we show that for object lying near the mirror's center of curvature or its vertex the angles between optic axis and the incident light rays can take arbitrary values, that is, the usual paraxial condition can be relaxed in these cases. Next, we consider the refracting ball and show that there can occur the situations, where the paraxial model is unable to describe the behavior of the refracted rays even qualitatively (the biconvex lens with $n > 1$ is a diverging lens for all or some rays).

V. REFERENCES

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