ALTERNATIVE METHOD FOR THE PHYSICAL INTERPRETATION OF THE NUT SOLUTION

MÉTODO ALTERNATIVO PARA LA INTERPRETACIÓN FÍSICA DE LA SOLUCIÓN NUT

L. A. López[a], O. Pedraza[a], V. E. Ceron[a], R. Arceo[b†]

a) Área Académica de Matemáticas y Física, Universidad Autónoma del Estado de Hidalgo, Carretera Pachuca-Tulancingo Km. 4.5, C. P. 42184, Mineral de la Reforma, Hidalgo, México

b) Facultad de Ciencias en Física y Matemáticas, Universidad Autónoma de Chiapas, C. P. 29050, Tuxtla Gutiérrez, Chiapas, México; roberto.arceo@unach.mx[†]

† corresponding author

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An alternative analysis of the NUT solution's physical interpretation is carried out, using the rod structure method of axial-symmetric solutions in Weyl coordinates. We obtain that the NUT solution can be interpreted as two counter-rotating rods of infinite rotation and with identical masses and a central static rod of different mass.

Se realiza un análisis alternativo de la interpretación física de la solución NUT, por medio del método de la estructura de barras de soluciones axiales simétricas en coordenadas de Weyl. Obteniendo que la solución NUT se puede interpretar como dos barras contra rotantes de rotación infinita y con masas idénticas y una barra central estática de masa diferente.

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I. INTRODUCTION

The Taub-NUT solution, first derived by Taub (1951) [1] and then by Newman et al. (1963) [2], has been the subject of intensive study due to its interesting properties. For example the NUT parameter contained in the solution has been related to the force of the gravomagnetic monopole, but its interpretation is under debate. The NUT solution is stationary, asymmetric, but not globally asymptotically flat because it has a semi-infinity singularity on the axis of symmetry at $\theta = \pi$. In [3] this singularity was interpreted as a semi-infinite massless source endowed with a finite angular momentum. The NUT solution has also been shown to be relevant in studies of black hole entropy for binary solutions [4,5].

In [6] proved that all geodesics of NUT space lie on spatial cones; this property also leads to gravitational lensing. In [7], the author proposes the construction of Skyrme fields; the procedure is implemented for Atiyah–Hitchin and Taub–NUT instantons. The NUT parameter has been implemented to study the stationary axial-symmetric space-time and nonlinear Born-Infeld electrodynamics [8].

A physical interpretation of the solution was given in [9], where they show that the NUT solution is interpreted as two counter-rotating semi-infinite sources of negative mass and infinite angular momentum, and between them a finite static source of positive mass. This study is carried out writing the solution in terms of the potentials of the Erns [10] and the integrals of Komar [11]. These techniques are widely used to obtain and analyze binary solutions.

Stationary axial-symmetric solutions can also be characterized by means of the rod structure, which provides information about the sources, which make up the complete source, and can be an alternative method to give a physical interpretation of the solutions, as can be seen in [12] and [13], that is why it is proposed to make one of said method, to study the NUT metric, this in order for the reader to observe how to use the rod structure. Given that the physical interpretation has already been given in [9], it can be seen that when using the rod structure we obtain the same interpretation but with an alternative method and more simple. This type of solutions could serve as support material in some gravitation course and it can also be the starting point or motivation for the students interested in the area. This work can be useful to the scholars in the area of mathematical methods for gravitation.

In section II the stationary axial-symmetric Weyl metric is briefly explained, later in section III an analysis is made of the solution in order to be characterized by the rod structure. Then in section IV it is mentioned how to obtain the information of mass and angular momentum of the rod face that makes up the rod structure of the solution. In section V the behavior of the NUT metric is studied by analyzing the rod structure, obtaining that the behavior is similar to the one reported in [9]. Finally in VI general conclusions are given.

II. WEYL'S METRIC

There are different ways to write the solutions of Einstein's equations in vacuum, depending on their symmetry; A stationary axial-symmetric solution can support two Killing vector fields V_1 and V_2, that generate a group G_2, the group is an abelian group if the Killing vectors commute. Almost all known interesting four-dimensional solutions of Einstein equations in vacuum, electrovacuum, or with some fundamental matter fields belong to this class G_2, and are known as the G_2 solutions. The line element of the G_2 solutions can be written in the form of Papapetrou (1966) [14],

$$ds^2 = e^{-2U}(g_{ab}dx^a dx^b + \alpha^2 d\phi^2) - e^{2U}(dt + Wd\phi)^2. \quad (1)$$

Where we consider the Killing vectors $V_1 = \partial_t$ and $V_2 = \partial_\phi$, and the metric functions U, g_{ab}, W and α depend only on the coordinates $x^a = (x^1, x^2)$. The isotropic coordinates are obtained from considering $\alpha = \rho$, they are known as Weyl coordinates $(x^1 = \rho, x^2 = z)$.

In 1917 Weyl [14] found a 4-dimensional (4D) static axial-symmetric $(W = 0)$ solution of Einstein's field equations in vacuum, which is given by:

$$ds^2 = -e^{2U}dt^2 + e^{-2U+2\sigma}(d\rho^2 + dz^2) + \rho^2 e^{-2U}d\phi^2, \quad (2)$$

where $\sigma(\rho, z)$ satisfies,

$$\partial_\rho \sigma = \rho\left[(\partial_\rho U)^2 - (\partial_z U)^2\right], \quad (3)$$

$$\partial_z \sigma = 2\rho(\partial_\rho U)(\partial_z U), \quad (4)$$

where $U(\rho, z)$ is an arbitrary solution of Laplace's equation $U_{,\rho\rho} + \frac{1}{\rho}U_{,\rho} + U,zz = 0$, in a three-dimensional flat space with metric:

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2. \quad (5)$$

Since $U(\rho, z)$ does not depend on ϕ, it can be considered as an axial-symmetric potential, corresponding to a certain density of mass per unit length. For example, the Schwarzschild solution corresponds to taking the source for U to be a thin rod on the z-axis with mass 1/2 per unit length.

As the coordinates ρ and z can be chosen that span two-dimensional surfaces orthogonal to all the Killing vector fields ∂_{y^i} $(y^i = (t, \phi))$ and then extended along the integral curves of said fields. In this coordinate system, the vectors ∂_ρ and ∂_z are orthogonal to ∂_{y^i}. If it is further assumed that the Killing vector fields are orthogonal to each other, then the metrics (1) and (2) can be written in compact form canonical (for reference see [5]) or in Weyl coordinates such as,

$$ds^2 = \sum_{i,j=1}^{2} G_{ij}dy^i dy^j + e^\sigma(d\rho^2 + dz^2), \quad (6)$$

with $\rho = \sqrt{|\det(G_{ij})|}$, where G_{ij} and σ are functions only of ρ and z. Considering Einstein's equations in vacuum $R_{ab} = 0$, we obtain that the equations for σ are,

$$\partial_\rho \sigma = -\frac{1}{2\rho} + \frac{\rho}{8}\sum_{i,j,k,l=1}^{2} G^{ij}G^{kl}\partial_\rho G_{ik}\partial_\rho G_{jl} - \frac{\rho}{8}\sum_{i,j,k,l=1}^{2} G^{ij}G^{kl}\partial_z G_{ik}\partial_z G_{jl},$$

$$\partial_z \sigma = \frac{\rho}{4}\sum_{i,j,k,l=1}^{2} G^{ij}G^{kl}\partial_\rho G_{ik}\partial_z G_{jl}. \quad (7)$$

III. STATIONARY SOLUTION ANALYSIS

In the stationary solution (6) it can be analyzed as in [5], $G_{ij} = G$ requires to be continuous; from the equation $\det G = \rho^2$, However, this breaks down as $\rho \to 0$, because for $\rho = 0$ we have that $\det G = 0$ so G is not invertible. Also it is observed that the product of the eigenvalues of $G(\rho, z)$ goes to zero for $\rho \to 0$, the eigenvalues $G(0, z)$ are real, being G symmetrical and include the zero eigenvalue.

A necessary condition for a regular solution is that precisely an eigenvalue of $G(0, z)$ be zero for some given z. Briefly dim (ker $G(0, z)) \geq 1$ except for isolated points denoted as $a_1, a_2, a_3, ..., a_N$ with $a_1 < a_2 < a_3 <, ..., < a_N$. So the axis z is divided into intervals $(-\infty, a_1)$, $(a_1, a_2), .., (a_N, \infty)$ known as $N + 1$ intervals or rods of the solution.

The solution G_{ij} has $(N+1)$ rods (a_{k-1}, a_k) with $k = 1, 2, ..., N+1$, defining for the solution G, $(N + 1)$ vectors v_k in $\Re^2$ as,

$$G(0, z)v_k = 0 \quad \text{for} \quad z \in (a_{k-1}, a_k), \quad k = 1, ..., N + 1. \quad (8)$$

If $v_k \neq 0$, that is, $v_k \in Ker[G(0, z)]$, v_k is called the direction of the corresponding rod (a_{k-1}, a_k). The rod structure of the solution G is defined as the specification of the intervals (a_{k-1}, a_k) plus the corresponding directions v_k related with the Killing vectors V_1 and V_2. Let us mention that v_k exists and is unique.

In general, the rods can be characterized as follows [15]:

- Finite rods located in the temporaloid direction ∂_t correspond to event horizons in space weather semi-infinite rods in ∂_t correspond to acelerated horizons.

- Rods located in the spacial directions ∂_ϕ correspond to intervals in the orbit of $\partial/\partial\phi$; if the rod is semi-infinite, then this set extends to infinity, corresponding to an axis of rotational symmetry with ϕ acting as the azimuthal angle. The ϕ must be identified with a certain period but it is related to conic singularities.

The rod structure provides a tool to analyze stationary solutions, although it is not possible to characterize to a solution due to its rod structure, since there can be different solutions that contain the same structure of rods. To give an unique characterization, in addition to the rod structure, it is necessary to impose conditions such as flat nod (see for example [16]).

IV. MASS AND MOMENT ASSOCIATED WITH THE ROD STRUCTURE

To interpret the parameters found in the metric functions of the solution (6), we will follow the analysis of T. Harmark and P. Olesen [17]. For this, the following quantity is defined,

$$\overrightarrow{C}(\rho, z) = \overrightarrow{C} = G^{-1}\overrightarrow{\nabla}G, \quad (9)$$

where $G(\rho, z)$ is given in (6) and (7). The components of the new element are: $C_\rho = G^{-1}\partial_\rho G$ and $C_z = G^{-1}\partial_z G$, so (9), it complies with the equation $\overrightarrow{\nabla} \cdot \overrightarrow{C} = 0$ for $\rho > 0$. In $\rho = 0$, we can have sources and it is possible to determine a source if the following equation is fulfilled:

$$\overrightarrow{\nabla}\cdot\overrightarrow{C} = 4\pi\delta^2\rho(z). \quad (10)$$

The delta-function δ^2 expresses that we have sources for $\overrightarrow{C}$ at $\rho = 0$. Since $\overrightarrow{C}$ complies with Gauss's law-like equation (10).

Now we propose that $\overrightarrow{C}$ (matrix $2x2$) as function a potential $B(\rho, z)$ as;

$$C_\rho = -\frac{1}{\rho}\partial_z B(\rho, z); \quad C_z = -\frac{1}{\rho}\partial_\rho B(\rho, z). \quad (11)$$

When we consider a cylindrical volume $V = (\rho \leq \rho_0, z_1 \leq z \leq z_2)$ and Gauss' law for $\overrightarrow{C}$:

$$\int\limits_V \overrightarrow{\nabla}\cdot\overrightarrow{C}dV = \int\limits_S \overrightarrow{n}\cdot\overrightarrow{C}dS \quad (12)$$

we obtain that

$$-\int\limits_0^{\rho_0}\partial_\rho B|_{z_1}d\rho + \int\limits_0^{\rho_0}\partial_\rho B|_{z_2}d\rho - \int\limits_{z_1}^{z_2}\partial_z B|_{\rho_0}dz = B(0,z_1) - B(0,z_2) \quad (13)$$

and perform an analysis, it is possible to obtain a matrix that represents the density of the system source, by means of $\rho(z) = -\frac{1}{2}\partial_\rho B(0,z)$ or $\rho(z) = \frac{1}{2}\lim\limits_{\rho\to 0}\rho C_\rho$.;

In the case the of solutions asymptotic to Minkowski space the behavior of $B(\rho, z)$ (see the work of T. Harmark and P. Olesen [17]) are,

$$B_{11}(\rho, z) = -\frac{2Mz}{\sqrt{\rho^2+z^2}} \qquad B_{12}(\rho, z) = \frac{2Jz(3\rho^2+2z^2)}{(\rho^2+z^2)^{3/2}} \quad (14)$$

Being M the mass of the rod and J the angular momentum, with the help of (13), $\rho(z)$ and the behavior of $B(0,z)$ (14) we have,

$$2M = \int\limits_{z_1}^{z_2} dz\rho_{11}(z), \qquad -4J = \int\limits_{z_1}^{z_2} dz\rho_{12}(z), \quad (15)$$

Thus, with the help of both matrices, it is possible to include all the information of the rods (direction, cut and density or mass) of the solution to be studied.

V. NUT METRIC

The NUT metric in Weyl coordinates takes the form,

$$ds^2 = e^{-2U}\left[e^{2K}(d\rho^2+dz^2) + \rho^2 d\phi^2\right] - e^{2U}(dt + Ad\phi)^2, \quad (16)$$

where

$$e^{2U} = \frac{(r_+ + r_-)^2 - 4(m^2+l^2)}{(r_+ + r_- + 2m)^2 + 4l^2}; \quad A = \frac{l}{\sqrt{m^2+l^2}}(r_+ - r_-), \quad (17)$$

where $r_\pm^2 = \rho^2 + (z \pm a_0)^2$ with $a_0 = \sqrt{m^2+l^2}$. Being m the parameter associated with the mass of the linear source and l the so-called NUT parameter.

Carrying out the analysis of the rod structure, we notice that the NUT metric has the form of the metric (6), so we have,

$$G(\rho, z) = \begin{pmatrix} -e^{2U} & -e^{2U}A \\ -e^{2U}A & \rho^2 e^{-2U} - e^{2U}A^2 \end{pmatrix}, \quad (18)$$

and in the $\rho \to 0$

$$G(0, z) = \begin{pmatrix} -e^{2U_0} & -e^{2U_0}A_0 \\ -e^{2U_0}A_0 & -e^{2U_0}A_0^2 \end{pmatrix}. \quad (19)$$

As mentioned in the previous section, the values that divide the axis z $(-\infty, \infty)$, are those that do not comply with dim (ker $G(0, z)) \geq 1$, and in the case of NUT, it is easy to see that they are $z = \pm a_0$, since $G(0, \pm a_0) = 0$, so we will have the next rods $(-\infty, -a_0)$, $(-a_0, a_0)$, (a_0, ∞).

V.1. *Rod Analysis* $(-\infty, -a_0)$

If we consider values of $z \in (-\infty, -a_0)$ we will have the following relationships $z - a_0 < 0$ and $z + a_0 < 0$, which applying to the metric functions, we would have $A_0 = \pm 2l$ and $e^{2U_0} = \frac{z^2 - a_0^2}{(mz)^2 + l^2}$, taking the equation (8), $G(0, x)v_k = 0$, with $v_k = (v_t, v_\phi)$. Carrying out the analysis, we obtain the relation $v_t + \frac{A}{2}v_\phi = 0$, so we can take $v_t = 1$ and $v_\phi = -\frac{2}{A_0}$, that is, the rod $(-\infty, -a_0)$ has address (v_t, v_ϕ).

V.2. *Rod Analysis* $(-a_0, a_0)$

If we consider values of $z \in (-a_0, a_0)$ we will have the following relationships $z + a_0 > 0$ and $z - a_0 < 0$, that applying to the metric functions it is obtained that $A_0^2 = \frac{l^2}{a_0^2}4z^2$ and the element $G_{\phi\phi} = (\rho^2 e^{2U} - e^{2U}A^2) \to -\frac{z^2 - a_0^2}{2a_0}$ in $\rho \to 0$. Carrying out the corresponding analysis, we have that the rod $(-a_0, a_0)$ has an address only in v_t.

V.3. *Rod Analysis* (a_0, ∞)

The analysis of the rod (a_0, ∞) is very similar to the rod $(-\infty, -a_0)$, in this case it is obtained that the rod also has the address $v_k = (v_t, v_\phi)$, but with $v_t = 1$ and $v_\phi = \frac{2}{A_0}$.

As for the rod structure of the NUT solution, it can be visualized as in the figure (1). Where it is observed how the axis z is divided into intervals, each rod has a certain mass M_i and momentum J_i, it is also shown that in the case of the intervals $(-\infty, -a_0)$ and (a_0, ∞) have two addresses ∂_t and ∂_ϕ, which is represented by the dotted lines between the rods, and the arrow indicate that they are counter-rotating. Also, the figure (1) shows the directions of symmetry (∂_t and ∂_ϕ)

related to Killing vectors, where the rods or regions of the solution are represented.

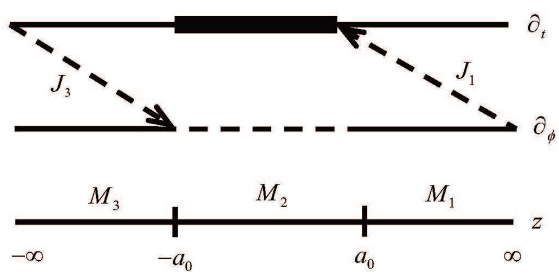

Figure 1. NUT metric rod structure showing directions ∂_t and ∂_ϕ.

Finally the mass and angular momentum of each rod can be obtained using the equations (15), for the rod (a_0, ∞) it will be obtained,

$$2M_1 = \int_{a_0}^{\infty} \frac{2l^2(a_0^2 - z^2)}{(l^2 + (m+z)^2)^2} dz = m - \sqrt{l^2 + m^2}, \tag{20}$$

analyzing now the angular momentum we will have,

$$-4J_1 = \int_{a_0}^{t} -\frac{4l[-2a_0^2 l^2 + l^4 + (m+z)^4 + 2l^2(m^2 + 2mz + 2z^2)]}{[l^2 + (m+z)^2]^2} dz$$
$$= \frac{4l(m+t)(l^2 + m^2 - t^2)}{l^2 + (m + t^2)}; \qquad J_1 = \lim_{t\to\infty} lt. \tag{21}$$

The same analysis is applied to the rod $(-\infty, -a_0)$ obtaining that $M_3 = M_1$ and $J_3 = -J_1$.

Now we analyze the rod $(-a_0, a_0)$, obtaining $J_2 = 0$ and that the mass of the rod is $M_2 = \sqrt{m^2 + l^2}$. It is very easy to observe that from the sum of the masses of the three rods the mass of the linear source m is obtained, and that the solution has momentum $J = 0$. Remember that the rod structure is a way of observing behaviors of axial-symmetric solutions. What was obtained agrees with what was studied by V. S Manko and E. Ruiz [9], in this work they used another method, such as Komar integrals, obtaining the same information. In other words, the NUT solution can be physically interpreted as the model of two counter-rotating sources at the ends and a source in the center with static positive mass and semi-infinite rods with negative masses.

VI. CONCLUSIONS

By studying the rod structure of the NUT solution in Weyl coordinates, it is possible to give an interpretation of its behavior. The NUT solution is interpreted as two infinite sources with opposite angular momenta, and negative mass and placed between them a finite static source with positive mass.

It is worth mentioning that the results obtained in this work are in agreement with the results obtained by V. S. Manko and E. Ruiz [9], because we obtained the same two semi-infinite regions (rods) with opposite infinite angular momenta and negative masses, also between the semi-infinite regions there is a finite region (rod) with the same mass reported by V. S. Manko and E. Ruiz.

However, the rod structure method is easy and suitable to use to characterize symmetric axial solutions of Einsten's equations that are in Weyl coordinates, as in this case, it was possible to apply it to the NUT solution even though it is a complex solution to be interpreted. With the rods method, the Nut solution's analysis allows to determine the directions of symmetry where the rods or regions of the solution are found.

The intention of showing the application of the rod structure in this work is to provide alternative methods to interpret this type of solutions, and thus to have more tools to study and characterize solutions of Einstein's equations with axial symmetry.

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