RAY PATHS THROUGH A GRIN LENS: THE CRYSTALLINE CASE

PASO DE LA LUZ A TRAVÉS DE UNA LENTE GRIN: EL CASO DEL CRISTALINO

R. C. Cruz-Rodríguez[a], A. L. Batista-Planas [a], O. Núñez-Chongo[a], C. Muñoz-Villaescusa[a] y A. J. Batista-Leyva[a,b†]

a) Instituto Superior de Tecnologías y Ciencias Aplicadas, Quinta de los Molinos, Av Salvador Allende esq. Luaces. La Habana 10400, Cuba
b) Group of Complex Systems and Statistical Physics. Physics Faculty, University of Havana, 10400 Havana, Cuba, abatista@instec.cu†
† corresponding author

Recibido 10/10/2015; Aceptado 7/11/2015

Rays paths follow a complex trajectory through the human crystalline. This is due to the changes in refractive index with position: crystalline is a GRIN lens. To calculate these trajectories approximate methods are often employed. In this contribution our aim is to compare two numerical methods: the first one based in solving the vector differential equation of the ray paths, while the second one is based on Fermat's principle. For each method different numeric schema are applied, and the results compared based on precision and computing easiness. We found that the most efficient procedure is a Runge-Kuta algorithm with adaptive step for integrating the differential equation derived from Fermat´s principle. This procedure will be applied in a ray tracing computer program and also in an optimization algorithm to determine the refraction index distribution inside crystalline.

La luz sigue una trayectoria compleja al atravesar el cristalino. Esto se debe al cambio del índice de refracción con la posición: el cristalino es una lente GRIN. Para calcular dicha trayectoria con frecuencia se emplean métodos aproximados. El objetivo de este artículo es comparar dos métodos: el primero basado en la solución de la ecuación diferencial del rayo y el segundo basado en el principio de Fermat. En cada caso se emplean dos esquemas numéricos para resolver la ecuación diferencial correspondiente. Los resultados son comparados teniendo en cuenta la exactitud y el costo computacional. Se determinó que el método más eficiente es el basado en la resolución de la ecuación derivada del principio de Fermat mediante un esquema de Runge-Kutta de paso adaptativo. Este método será usado en un programa informático de trazado de rayos que se encuentra en desarrollo y como parte de un algoritmo de optimización para determinar los parámetros que caracterizan la distribución de índice de refracción en cristalinos.

PACS: Geometrical optics 42.15.-i, Gradient index (GRIN) devices 42.79.Ry, Ophthalmic optics 42.66.Ct, Eye 42.66.-p, Numerical methods (mathematics) 02.60.-x

I. INTRODUCTION

The mechanism of vision in humans is a rather complex process that includes the coordinated work of cornea, crystalline and controlling muscles, in order to form an in focus image on the retina (see fig. 1).

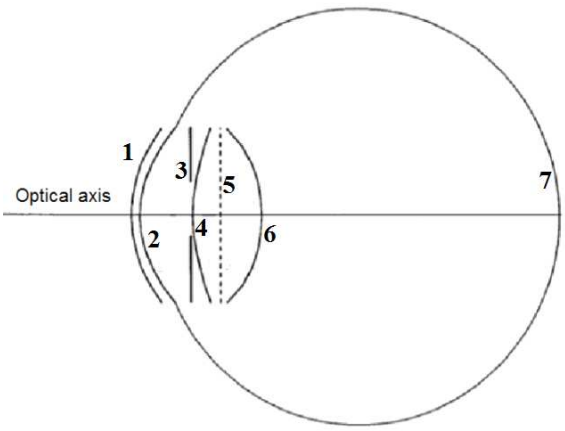

Figure 1. Schematic of human eye: anterior and posterior corneal surfaces (1, 2), iris (3), anterior and posterior crystalline surfaces (4, 6), plane containing crystalline nucleus (5), retina (7).

Particularly the crystalline plays a key role, firstly through accommodation, i. e. the variation of its focal length due to changes in thickness and curvature of its surfaces [1]. But also the peculiar density distribution of proteins both in radial and axial directions creates a gradient of the refraction index: crystalline is a GRIN lens [2, 3]. This fact makes difficult the exact calculation of ray paths through crystalline.

An additional problem is related with difficulties in the determination of the distribution of the refraction index. Firstly, because the study of in vivo refraction index distribution is only attainable using MRI [4]. Secondly, because GRIN's parameters change with age [5]. This provokes that frequently the GRIN distribution is substituted in calculations by an average refraction index [6, 7] with the correspondingly lost in accuracy. So, a method for determining the distribution function is of paramount importance. There is an indirect method of measuring this distribution in vitro. Let us suppose that we measure the incoming and outgoing directions of a light ray in an isolated crystalline. Then, using an optimization algorithm, we could find the distribution of refraction index that better reproduces the experimental details. Clearly, for doing this we need a method to calculate the trajectory knowing the index distribution. Though there are methods to make the analytical tracing of light rays in the paraxial region [8], approximate methods are more often employed.

A widely used approach to calculate ray paths through human lens (due to its simplicity) is to consider a shell model [2,3,9]. The continuous index distribution is substituted by a

discrete increment of the refraction index from the surface to the core, dividing the crystalline in shells, each one having a constant value of refraction index, equal to that of its outer surface. This model reproduces well the overall behavior of human lens [9] when there is no need of great accuracy. Another approach, computationally more expensive, but also more exact, is to solve a differential equation describing the ray trajectories. Here we consider different numeric schema for a widely applied GRIN model named bi parabolic model [10] used by some of us in a previous work [11].

We solved the model by two different methods: firstly using the eikonal equation and secondly with the aid of Fermat´s principle. In both cases the resulting differential equation is solved numerically, and all results are compared. The main objective of the present contribution is to evaluate which of the different numerical approaches is more suited for determining the ray path through the crystalline given an index distribution function. The selected method will be applied in solving two important problems: the determination of the index distribution in a crystalline from experimental data, and its use in a computational ray tracing model of human eye to be applied in ophthalmologic practice.

II. THEORY

II.1. GRIN MODELS

Crystalline is a GRIN lens that varies its refraction index both in axial and transversal directions. Most authors agree in considering a monotonic increment of refraction index from the surface to the nucleus [13]. This increment has been modeled in different ways, among which the more used are:

- A bi parabolic model [10], It considers the dependence of the refraction index with position as:

$$n(z,w) = n_{00} + n_{01}z + n_{02}z^2 + n_{12}w \tag{1}$$

where z is the coordinate along the symmetry axis of the lens and $w^2 = x^2 + y^2$ is perpendicular to it. n_{ij} are parameters of the model, adjusted from experimental data. Due to the fact that crystalline refraction index has a maximum at its nucleus, and a minimum at the outer surface, the model can be implemented in the following way:

$$n(z,w) = \begin{cases} n_n + n_{a1}(z-d)^2 + n_0 w^2 \\ n_n + n_{p1}(z-d)^2 + n_0 w^2 \end{cases} \tag{1.1}$$

Here d is the z coordinate of the nucleus, which is contained in plane (5), fig. 1. The model depends on 5 parameters.

- The 3-parameter model:

$$n(\rho,\theta) = n_n + (n_n - n_s)\left[\frac{\rho}{\rho_s(\theta)}\right]^p \tag{2}$$

where (ρ,θ) are the polar coordinates of a point inside crystalline, the sub index n refers to the nucleus and s to the surface of the crystalline, $\rho_s(\theta)$ is the distance from the nucleus to the surface in θ direction. p is the GRIN coefficient that has been reported to vary with age and is around 3 [12].

For computational easiness we will use here the first one. We plan to use the other one in a future work.

II.2. SOLVING THE GRIN LENS

To calculate the ray paths through crystalline (independently of the GRIN model used) we apply two strategies: in the first one we use the vector differential equation of the ray paths, derived from the eikonal equation [14]:

$$\frac{d}{ds}\left[n(s)\frac{d\vec{r}}{ds}\right] = \nabla n(s) \tag{3}$$

In eq. 3 is the position vector of a point in the ray path, s is the length of the ray from an appropriate origin, and n(s) gives the variation of refraction index along the trajectory. Expanding the square brackets and introducing the tangent vector eq. 3 transforms into the following system of differential equations:

$$\frac{d\vec{r}}{ds} = \vec{t}(s) \tag{4a}$$

$$\frac{d\vec{t}}{ds} = \frac{1}{n(s)}\left[\nabla n(s) - \frac{dn}{ds}\vec{t}\right] \tag{4b}$$

Introducing the dependences with (z,w) and a new variable τ, defined as $d\tau = ds/n(s)$, we finally find:

$$\frac{dz}{d\tau} = t_z(\tau) \tag{5a}$$

$$\frac{dw}{d\tau} = t_w(\tau) \tag{5b}$$

$$\frac{dt_z}{d\tau} = n(z,w)\frac{\partial n}{\partial z} \tag{5c}$$

$$\frac{dt_w}{d\tau} = n(z,w)\frac{\partial n}{\partial w} \tag{5d}$$

Equations 5a-5d can be solved with an appropriate numerical method if the functional dependence of the GRIN model is known. For this, proper initial condition has to be imposed.

The second method is based on Fermat's principle, which states (in its strongest formulation [15]) that the optical length of a real ray between any two points is shorter than the optical length by any other curve which joins the two points and lies in a certain regular neighborhood of it. The optical path is defined as:

$$\Delta l = \int_a^b n(s)ds \tag{6}$$

It is easy to verify that the principle also states that the time the light travels by the actual trajectory is smaller than the time by any other possible trajectory. Writing $n(s)$ and the

differential path ds in a reference frame, for instance (z, w), we obtain:

$$\Delta l = \int_a^b n(z,w)\sqrt{1+[w'(z)]^2}dz = \int_a^b L(z,w)dz \tag{7}$$

This variational principle takes us to the Lagrange–Euler differential equation:

$$\frac{\partial L}{\partial w} - \frac{d}{dz}\frac{\partial L}{\partial w'} = 0 \tag{8}$$

Operating eq. 8 it yields [11]:

$$w''(z) = f_1\left\{1+[w'(z)]^2\right\} - f_2\left\{w'(z)+[w'(z)]^3\right\} \tag{9}$$

where,

$$f_1 = \frac{1}{n(z,w)}\frac{\partial n(z,w)}{\partial w};\ f_2 = \frac{1}{n(z,w)}\frac{\partial n(z,w)}{\partial z} \tag{10}$$

For the particular case of eq. 1 we obtain:

$$f_1 = \frac{2n_{12}w}{n(z,w)};\ f_2 = \frac{n_{01}+2n_{02}z}{n(z,w)} \tag{11}$$

Equation (9) can be reduced to a first order differential equation defining $m(z) = w'(z)$, to obtain:

$$\frac{dw}{dz} = m(z) \tag{12a}$$

$$\frac{dm}{dz} = \frac{1}{n(z,w)}\left[1+m^2(z)\right]\left[\frac{\partial n}{\partial w} - \frac{\partial n}{\partial z}m(z)\right] \tag{12b}$$

Equations 12a-12b can be solved numerically.

III. NUMERICAL PROCEDURES

Numerical integration of the equations 5a-5d and 12a-12b was performed using a fifth-order embedded Runge–Kutta (RK45), found in reference [16] which is based on a paper by Dormand and Prince [17]. Dormand–Prince method is an adaptive step size control method to solve the general problem of the form:

$$\frac{dy}{dx} = f(x,y) \tag{13a}$$

$$y(x_0) = y_0 \tag{13b}$$

in the discrete set of points (x_i, y_i).

The adaptive step size control means that to approximate the solution x_f, y_f the equation 13a is integrated advancing from x_0 to x_f with a variable step size $h_i = x_i - x_(i-1)$ and h_i are selected maintaining a prescribed error tolerance for the numerical solution y_f with the minimum computational effort. This feature is strongly desired when evaluating the function $f(x_i, y_i)$ (in our case depends on $n(z,w)$ model) is computational expensive.

The equations where also solved using classical Runge-Kutta 4 (RK4) method (Sharma formula for equations 5a-5d). Equations 5a-5d and 12a-12b need to be completed with the initial conditions in order to write them in the form of 13a-13b and solve numerically.

For 5a-5d the initial conditions can be written as:

$$z(0) = z_0 \tag{14a}$$

$$w(0) = w_0 \tag{14b}$$

$$t_z(0) = n(z_0, w_0)t_{z0} \tag{14c}$$

$$t_w(0) = n(z_0, w_0)t_{z0} \tag{14d}$$

where $[t_{z0}, t_{w0}]$ is a unitary vector related to m_0 (the direction of the ray on the initial point (z_0, w_0) by the equations:

$$t_{z0} = \frac{1}{\sqrt{1+m_0^2}} \tag{15a}$$

$$t_{w0} = \frac{m_0}{\sqrt{1+m_0^2}} \tag{15a}$$

For 12a-12b we have:

$$w(z_0) = w_0 \tag{16a}$$

$$w'(z_0) = m_0 \tag{16b}$$

The outgoing point at the posterior crystalline surface was calculated using Hermit interpolation between last point inside the crystalline and first point outside, so it was needed to test if the ray path was inside the crystalline in each iteration step.

Due to the fact that the GRIN model does not have analytical solutions, we established a numerical standard to compare with. This was calculated using a computationally expensive Runge Kutta 8, with adaptive step (RK853) [18] with high precision arithmetic (float128 type as defined in the Boost library of C++, which guaranties 34 exact decimal digits). This method is too slow for being used in real applications, but is exact enough for using it as comparison standard. The combination RK853 – float128 allows to obtain more than 16 exact significant digits, which can be considered exact if you compare with a type double data in C++. All programs were implemented in this language.

IV. RESULTS AND DISCUSSION

In what follows, we will solve the equations derived from the eikonal equation and from Fermat's principle with the two numerical methods explained above. Their results will be labeled as E-RK4, E-RK45 and F-RK4, F-RK45 respectively. The comparison parameter, the absolute error (ϵ), is the modulus of the difference between the value obtained with a given method and the one obtained with RK853.

As comparison parameters were selected:

- the number of evaluations necessary to get a given absolute error in the position of the exit ray.

- the number of evaluations necessary to get a given absolute error in the slope of the exit ray.
- the processing time necessary to get a given absolute error in the position and slope of the exit ray for a given computer configuration.

Table 1. Parameters of the GRIN model

d [mm]	n_N	n_{a1}	n_{p1}	n_0
1.59	1.407	-0.015427	-0.006605	-0. 001978

Table 1 shows the parameters of the model, used in eq. (1').

We describe the crystalline as the locus between two aspheric surfaces (anterior and posterior) with equations:

$$w^2 + (Q+1)z^2 - 2zR = 0 \tag{17}$$

In eq. 17 Q is the asfericity parameter, and R is the surface radius at the center of symmetry. The geometrical data describing crystalline surfaces were taken from ref. [11] and are shown in Table 2.

Table 2. Geometric parameters of the crystalline.

Surf	Asf (Q)	Radius R [mm]	Pos z [mm]
Ant	-0.94	12.4	0.00
Post	0.96	-8.1	4.02

Figure 2 displays the number of evaluations needed in each method in order to achieve a given absolute error in the position of the exit ray. As position we mean the w coordinate, because once we know this value, z coordinate is univocally determined from the equation of the posterior crystalline surface.

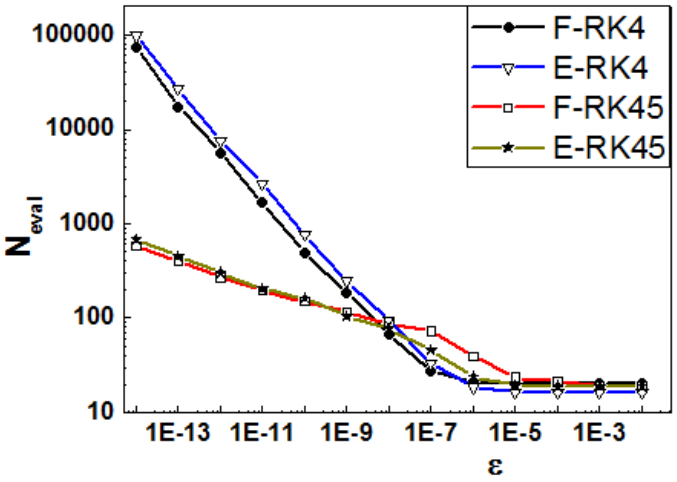

Figure 2. Number of evaluations needed in order to achieve a given error for each method: Position of the exit ray.

From fig. 2 it is easy to see that RK adaptive performs better if a high accuracy in the position is needed. This is true both for the eikonal and Fermat´s equations.

Regarding the precision in calculating the exit slope, fig. 3 shows results that are consistent with those of fig.2.

Summarizing, when using the numeric schema RK4 the number of evaluations increases very fast when a high precision is needed, though for errors above 10^{-7} the number of evaluations is smaller than the required for RK45. This is easy to understand: when the allowed tolerance is big, the six evaluations per step needed in RK45 surpass the four needed in RK4. But when the tolerance is small, the adaptive step compensates this difference, diminishing the number of iterations needed for RK45.

From figs. 2 and 3 it is evident that numeric procedure RK45 is more convenient for calculating the trajectories when high precision is needed. Regarding the selection of the most appropriate differential equation, though the number of evaluations for solving the equation derived from Fermat´s principle is always smaller than in the case of using the eikonal equation, the difference is not remarkable. So, for having a better selection criterion, we use another comparison quantity: execution time.

For this, we worked in a computer with the following architecture, which can be easily present in an average ophthalmological service in Cuba: Intel(R) Celeron(R) CPU E3400 at 2.60 GHz and 2 GB of RAM.

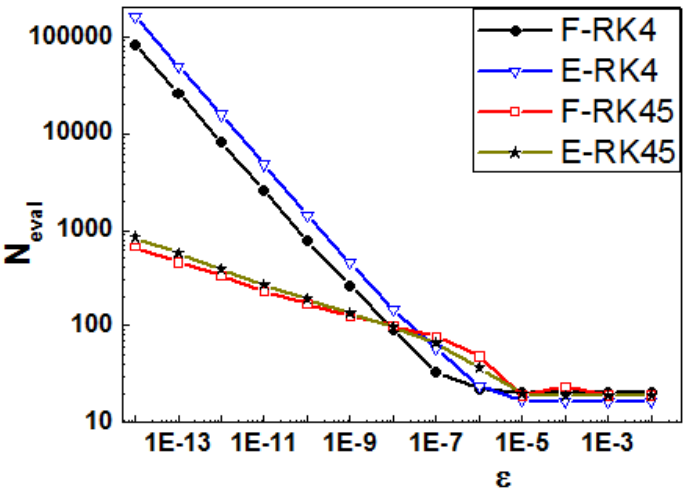

Figure 3. Number of evaluations needed in order to achieve a given error for each method: Slope of the exit ray.

Figure 4 shows the execution time, calculated as the average of 50 different tracks through crystalline for a given error: fig. 4a shows the execution time to achieve a given accuracy in the position of the exit ray and fig. 4b in the slope of the exit ray.

It is important to understand that these are the most important parameters, because together they determine the position at which the light ray intercepts the retina, and so the visual sensation.

Figure 4 gives answer to the raised question: the execution time is smaller in the Fermat formulation. Though the difference seems to be too small to matters, if the model were used to reconstruct the image of an object, thousands of rays should be traced, and the difference becomes appreciable.

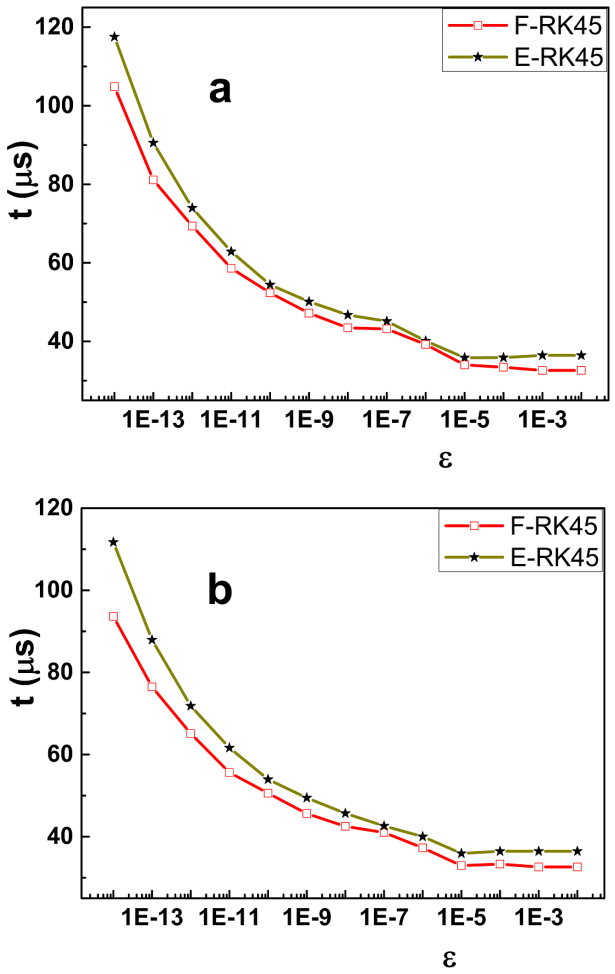

Figure 4. Execution time for achieving a given accuracy in the determination of the exit point (a) and the slope of the exit ray (b).

V. CONCLUSIONS

We have performed a group of numerical experiments in order to choose the best computer model for the determination of ray paths through a human crystalline. The main result is that a Runge Kuta integration algorithm with adaptive step applied to the differential equation derived from the application of Fermat's principle to the GRIN model of the human lens is the most efficient, and will be applied in the model eye under development by our research group. This will also be a part of an optimization algorithm for determining the GRIN parameters of the crystalline based on *in vitro* experiments.

This model will also help in the understanding of the influence of the lens in the aberrations of human eyes. In a future work we will apply this study to different models of GRIN, with polynomial higher order.

REFERENCES

[1] A. L. Sheppard et al, Invest. Ophthalmol. Vis. Sci. 52, 3689 (2011).

[2] M. A. Rama et al, Optics Communication 249, 595 (2005).

[3] M. V. Pérez et al, J. Opt. A: Pure App. Opt. 5, S293 (2003).

[4] C. E. Jones, D. A. Atchison and J. M. Pope, Optom. Vis. Sci. 84, 990 (2007).

[5] Y-L. Chen, L. Shi, J. W. L. Lewis and M. Wang, Opt. Express 20, 12498 (2012).

[6] C. de Freitas et al, Opt. Lett. 38, 85 (2008).

[7] W. N. Charmann and D. A. Atchison, Biomed. Opt. Express 5, 31 (2013).

[8] M. Bahrami and A. V. Gonchrov, J. Biomed. Opt. 17(5), 055001 (2012).

[9] Y. J. Liu et al, Optik 116, 241 (2005).

[10] H.-L. Liu and N. A. Brennan, J. Opt. Soc. Am. A 14(8), 1695 (1997).

[11] C. Muñoz-Villaescusa, O. Núñez-Chongo and A. J. Batista-Leyva, Rev. Cub. Fis. 31, 35 (2014).

[12] A. de Castro et al, J. Mod. Opt. 58, 1781 (2011).

[13] R. Navarro, J. Optom. 2(1), 3 (2009).

[14] M. Born and E. Wolf, Principles of Optics, 6th Ed. (Pergamon Press, New York, Oxford, 1986), pp 122 – 124.

[15] M. Born and E. Wolf, ibid, p 128.

[16] W. H. Press et al, Numerical Recipes on C. The Art of Scientific Computing, 2nd Ed. (Cambridge University Press, Cambridge, 1997), pp 731 – 777.

[17] J. R. Dormand and P. J. Prince. J. Comput. App. Math. 6, 19 (1980).

[18] J. R. Dormand and P. J. Prince. Runge-Kutta Triples. Computers and Mathematics with Applications, 12A, 1007 (1986).