DYNAMIC CAVITY METHOD IN FULLY-ASYMMETRIC MODELS

MÉTODO DE CAVIDAD DINÁMICO EN MODELOS COMPLETAMENTE ASIMÉTRICOS

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The physics of disordered systems is a broad and constantly evolving field. In this work we focus on the study of discrete variable models with asymmetric interactions, in particular the fully-asymmetric ferromagnet and the fully-asymmetric Sherrington-Kirkpatrick. We use the cavity master equation, a well-known technique for the out-of-equilibrium dynamics, to derive average equations describing the time evolution of the magnetization and the energy in these models. In this way, we recovered previous results for the magnetization known from the literature and obtained new equations for the energy. With this work, we contribute to establish the cavity master equation as one of the most relevant techniques in the study of out-of-equilibrium systems and clarify its relationship with previous methods.

La física de los sistemas desordenados es un campo amplio y en constante evolución. En este trabajo, nos centramos en el estudio de modelos de variables discretas con interacciones asimétricas, específicamente el modelo ferromagnético completamente asimétrico y el modelo Sherrington-Kirkpatrick completamente asimétrico. Utilizamos la ecuación maestra de cavidad, una técnica conocida para describir la dinámica fuera del equilibrio, para derivar ecuaciones promedio que describen la evolución temporal de la magnetización y la energía. De esta manera, recuperamos los resultados previos de la literatura sobre la magnetización y obtuvimos nuevas ecuaciones para la energía de estos modelos. Con este trabajo, contribuimos a asentar la ecuación maestra de cavidad como una de las técnicas más relevantes en el estudio de sistemas fuera de equilibrio y esclarecemos su relación con métodos previos.

Keywords: Cavity Master Equation (Ecuación maestra de cavidad), Discrete Variable Models (Modelos de variables discretas), Dynamic Cavity Method (Método de cavidad dinámico), Asymmetric Interactions (Interacciones asimétricas).

I. INTRODUCTION

The dynamic cavity method is a powerful tool in statistical physics and complex systems theory. It is used to derive both average equations and specific equations for individual graphs describing the macroscopic behavior of systems with many interactions. This method has proven to be particularly useful in the study of disordered systems [1–5]. Some methods that preced and influenced the dynamic cavity method are: the static-cavity method [6], the replica method [7], the dynamic mean-field theory (i.e. IBMF and PBMF) [8–10] and message-passing algorithms [11].

In this work we explore the application of the dynamic cavity method to obtain average equations in two specific models: fully-asymmetric ferromagnet [8] and the fully-asymmetric Sherrington-Kirkpatrick [12]. Specifically, in the dynamic cavity method, we use the cavity master equation (CME), first presented in [1]. These two models selected by us share two relevant qualities that make them more attractive. First, the asymmetry in the interactions allows us to propose suitable factorizations for the system's joint probability distribution, obtaining closed forms for the average equations. Second, we found previous analytic results in the literature in both cases [8, 9], which is a rare resource when studying the dynamics of disordered systems. Indeed, we insert this work in a field where few advances have been made over the years, and it is therefore important that we manage to connect a recently developed technique like the CME with known results.

We recover the equations obtained in [8] for the magnetization of the fully-asymmetric ferromagnet. In addition, we present here equations for the time evolution of the energy of the system not yet published and which were recently introduced in the Ph.D. thesis of [13]. On the other hand, from the same CME we re-derive the equations for the magnetization of the fully-asymmetric Sherrington-Kirkpatrick, already introduced in [9]. In the latter case, we obtain for the first time a set of equations for the time evolution of the system's energy.

The rest of this paper is organized as follows: first, we present the theoretical basis and how the cavity master equation is used in both models. Then, we analyze each model individually and show how to obtain averaged equations for the magnetization and energy from the cavity master equation. Finally, their prediction is compared with the results of numerical simulations.

II. THEORETICAL BASIS

In its first level of approximation, the cavity master equation is given as follows [1, 14]:

$$\frac{dp^t(\sigma_i \mid \sigma_j)}{dt} = -\sum_{\sigma_i'} \sigma_i \sigma_i' \, p^t(\sigma_i' \mid \sigma_j) \sum_{\sigma_{\partial i \setminus j}} r_i(\sigma_i', \sigma_j) \prod_{k \in \partial i \setminus j} p^t(\sigma_k \mid \sigma_i') \tag{1}$$

This equation is written for the continuous-time dynamics of a system with N discrete variables $\vec{\sigma} = \{\sigma_1, \ldots, \sigma_N\}$. The function r_i is the spin transition probability for σ_i, given the configuration of its neighbors. The dynamics is sequential or asynchronous, *, that is,*, we allow only one variable to change its value at each time t. In this case, we selected Glauber's dynamical rule [15]:

$$r_i(\sigma_i, \sigma_j, \sigma_{\partial i \setminus j}) = \frac{\alpha}{2}(1 - \sigma_i \tanh(\beta \sum_{k \in \partial i \setminus j} J_{ki}\sigma_k + \beta J_{ji}\sigma_j)) \tag{2}$$

where α provides a dynamical time scale, β is the inverse of the temperature, and the parameters J_{ij} are the couplings between interacting spins.

This was a special choice for running the simulations. However, they can be performed with any transition probability r_i that depends on the instantaneous values of the spins of the system. The pair probability equation is a particular case of the closure developed in Ref. [16]:

$$\begin{aligned} \frac{dP^t(\sigma_i, \sigma_j)}{dt} = &-\sum_{\sigma_i'} \sigma_i \sigma_i' \, P^t(\sigma_i', \sigma_j) \sum_{\sigma_{\partial i \setminus j}} r_i(\sigma_i', \sigma_j) \prod_{k \in \partial i \setminus j} p^t(\sigma_k \mid \sigma_i') \\ &-\sum_{\sigma_j'} \sigma_j \sigma_j' \, P^t(\sigma_i, \sigma_j') \sum_{\sigma_{\partial j \setminus i}} r_j(\sigma_j', \sigma_i) \prod_{k \in \partial j \setminus i} p^t(\sigma_k \mid \sigma_j') \end{aligned} \tag{3}$$

Taking a marginal of the Eq. (3) we obtain the individual probabilities $P(\sigma_j)$:

$$\begin{aligned} \frac{dP^t(\sigma_j)}{dt} = &-\sum_{\sigma_j'} \sigma_j \sigma_j' \sum_{\sigma_i} P^t(\sigma_i, \sigma_j') \times \\ &\times \sum_{\sigma_{\partial j \setminus i}} r_j(\sigma_j', \sigma_i) \prod_{k \in \partial j \setminus i} p^t(\sigma_k \mid \sigma_j') \end{aligned} \tag{4}$$

The Eq. (4) replaces the equation derived in [1]:

$$\begin{aligned} \frac{dP^t(\sigma_j)}{dt} = &-\sum_{\sigma_j'} \sigma_j \sigma_j' \sum_{\sigma_i} P^t(\sigma_j') \, p^t(\sigma_i \mid \sigma_j') \times \\ &\times \sum_{\sigma_{\partial j \setminus i}} r_j(\sigma_j', \sigma_i) \prod_{k \in \partial j \setminus i} p^t(\sigma_k \mid \sigma_j') \end{aligned} \tag{5}$$

As can be seen the Eq. (5) has a factor $P^t(\sigma_j') \, p^t(\sigma_i \mid \sigma_j')$ while the Eq. (4) has $P^t(\sigma_i, \sigma_j') = P^t(\sigma_j') \, P^t(\sigma_i \mid \sigma_j')$. This means that the Eq. (4) can be obtained from the Eq. (5) by replacing the conditional probability of the cavity $p^t(\sigma_i \mid \sigma_j')$ by the corresponding conditional probability $P^t(\sigma_i \mid \sigma_j')$.

III. FULLY-ASYMMETRIC FERROMAGNET

III.1. *Magnetization*

The key point of the following derivation is that, due to fully asymmetric interactions (unidirectional influence between variables), the probability distribution of the local field $h_i = \sum_{k \in \partial i} J_{ki}\sigma_i$ is independent of the corresponding spin σ_i. In models like this, where the sum $\sum_k J_{ki} J_{ik}$ vanishes in the thermodynamic limit, the Onsaguer reaction term [6] is not present and the spin σ_i has no effect on the field h_i.

The model couplings are obtained from the distribution:

$$Q(J_{ki}) = \frac{\lambda}{N-1}\,\delta(J_{ki} - 1) + (1 - \frac{\lambda}{N-1})\,\delta(J_{ki}) \tag{6}$$

We use this distribution to build the graph of interactions. For every possible pair (ik) in the system, J_{ki} and J_{ik} are drawn independently. Therefore, for finite λ and in the thermodynamic limit, the probability of $J_{ki} = J_{ik} = 1$ vanishes. This means that in the Eq. (4), given $J_{ki} = 1$, we know that $J_{ik} = 0$. So, the variables $p(\sigma_k \mid \sigma_j')$ in Eq. (1) do not really depend on σ_j'.

Using this and applying the operator $\sum_{\sigma_i} \sigma_i[\cdot]$ to the Eq. (4) yields:

$$\begin{aligned} \frac{dm_j(t)}{dt} = &-\alpha \, m_j(t) + \alpha \sum_{\sigma_j'} \sum_{\sigma_i} P(\sigma_i, \sigma_j') \times \\ &\times \sum_{\sigma_{\partial j \setminus i}} \tanh(\beta \sum_{k \in \partial j} J_{kj}\sigma_k) \prod_{k \in \partial j \setminus i} \Big(\frac{1 + \sigma_k \nu_k}{2}\Big) \end{aligned} \tag{7}$$

where we have made $p(\sigma_k \mid \sigma_j') \equiv p(\sigma_k) = \frac{1+\sigma_k \nu_k}{2}$ and we have defined $m_i(t) \equiv \sum_{\sigma_i} \sigma_i P(\sigma_i)$ and $\nu_i(t) \equiv \sum_{\sigma_i} \sigma_i p(\sigma_i)$. Explicitly doing the summation by σ_j' in the Eq. (7) and using $P(\sigma_i) = \frac{1+\sigma_i m_i}{2}$ we get:

$$\begin{aligned} \frac{dm_j(t)}{dt} = &-\alpha \, m_j(t) + \alpha \sum_{\sigma_{\partial j}} \tanh(\beta \sum_{k \in \partial j} J_{kj}\sigma_k) \times \\ &\times \Big(\frac{1 + \sigma_i m_i}{2}\Big) \prod_{k \in \partial j \setminus i} \Big(\frac{1 + \sigma_k \nu_k}{2}\Big) \end{aligned} \tag{8}$$

With this, we find the distribution of the local fields h_j acting on site j:

$$D(h_j) = \Big(\frac{1 + \sigma_i m_i}{2}\Big) \prod_{k \in \partial j \setminus i} \Big(\frac{1 + \sigma_k \nu_k}{2}\Big) \tag{9}$$

In fact, we could perform the summation over σ_j' because the distribution $D(h_j)$ is independent of σ_j' as mentioned above.

Analogously, we can get an equation for $\frac{d\nu}{dt}$:

$$\frac{d\nu_j(t)}{dt} = -\alpha\, \nu_j(t) + \alpha \sum_{\sigma_{\partial j}} \tanh(\beta \sum_{k\in\partial j} J_{kj}\sigma_k) \times \\ \times \Big(\frac{1+\sigma_i \nu_i}{2}\Big) \prod_{k\in\partial j\setminus i} \Big(\frac{1+\sigma_k \nu_k}{2}\Big) \qquad (10)$$

Therefore, if we choose an initial condition such that $m_i(0) = \nu_i(0)$ for all $i = 1, ..., N$, we have $m_i(t) = \nu_i(t)$ for all $t > 0$. This allows us to rewrite the Eq. (7) as:

$$\frac{dm_j(t)}{dt} = -\alpha\, m_j(t) + \alpha \sum_{\sigma_{\partial j}} \tanh(\beta \sum_{k\in\partial j} J_{kj}\sigma_k) \prod_{k\in\partial j} \Big(\frac{1+\sigma_k m_k}{2}\Big) \qquad (11)$$

We can average this equation over the graph ensemble given by the Eq. (6). The result is a special case of a result of Derrida *et al.* in [8]:

$$\frac{d\hat{m}(t)}{dt} = -\alpha\, \hat{m}(t) + \alpha e^{-\lambda}\, S(\hat{m}, \beta, \lambda, 0) \qquad (12)$$

where we define:

$$S(\hat{m}, \beta, q, \sigma) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \sum_{n=0}^{k} \binom{k}{n} \Big(\frac{1+\hat{m}}{2}\Big)^n \times \\ \times \Big(\frac{1-\hat{m}}{2}\Big)^{k-n} \tanh(\beta(2n-k+\sigma)) \qquad (13)$$

Now we will try to simplify this equation. Exchanging the sums and making the change of variables $l = k - n$, then $n' = n - l$ and substituting the modified Bessel function of first order:

$$S(\hat{m}, \beta, q, \sigma) = \sum_{n'=-\infty}^{\infty} \tanh(\beta(n'+\sigma)) \Big(\frac{1+\hat{m}}{1-\hat{m}}\Big)^{n'/2} \mathrm{I}_{n'}(q\, \sqrt{1-\hat{m}^2}) \qquad (14)$$

This shows that the probability of having a local field $\hat{h} = n$, which does not account for the connectivity of the node is:

$$D(\hat{h}{=}n) = \Big(\frac{1+\hat{m}}{1-\hat{m}}\Big)^{n/2} \mathrm{I}_n(\lambda\, \sqrt{1-\hat{m}^2}) \qquad (15)$$

III.2. Energy

In these out-of-equilibrium models, there is no Hamiltonian and, therefore, the word energy cannot have the traditional meaning. However, we used the concept of energy as a measure of the average intensity of the interactions between the spins.

If one wants to know how strongly two spins interact, it becomes necessary to extract information from the pair probabilities in Eq. (3). As with single-site probabilities, we can get the equation for average pair probabilities $\hat{P}_{J_{12},J_{21}}(\sigma_1, \sigma_2) \equiv \hat{P}_{J_1,J_2}(\sigma_1, \sigma_2)$:

$$\frac{d\hat{P}_{J_1,J_2}(\sigma_1,\sigma_2)}{dt} = -\frac{\alpha}{2} \sum_{\sigma} (\sigma\sigma_1 \hat{P}_{J_1,J_2}(\sigma,\sigma_2) + \sigma\sigma_2 \hat{P}_{J_1,J_2}(\sigma_1,\sigma)) \\ + \frac{\alpha\sigma_1}{2} \Big[\sum_{\sigma} \hat{P}_{J_1,J_2}(\sigma,\sigma_2)\Big] e^{-\lambda}\, S(\hat{m}, \beta, \lambda, J_2\sigma_2) \\ + \frac{\alpha\sigma_2}{2} \Big[\sum_{\sigma} \hat{P}_{J_1,J_2}(\sigma_1,\sigma)\Big] e^{-\lambda}\, S(\hat{m}, \beta, \lambda, J_1\sigma_1) \qquad (16)$$

Defining the energy as:

$$\hat{e}(t) = -\lambda \int dJ_1 dJ_2 Q_c(J_1, J_2)(J_1 + J_2) \times \\ \times \sum_{\sigma_1,\sigma_2} \sigma_1\, \sigma_2\, \hat{P}_{J_1,J_2}(\sigma_1, \sigma_2) \qquad (17)$$

For this definition of $\hat{e}(t)$ to make sense, we average over pairs (σ_1, σ_2) that are connected in the graph. In the large size limit, this means that $J_1 = 0$ and $J_2 = 1$ or $J_1 = 1$ and $J_2 = 0$. Therefore, we introduce the *connected* distribution $Q_c(J_1, J_2) = [\delta(J_1)\delta(J_2-1) + \delta(J_2)\delta(J_1-1)]/2$, which in practice is the joint probability distribution of (J_1, J_2) conditioned on one of them being nonzero. Therefore:

$$\hat{e}(t) = -\frac{\lambda}{2} \sum_{\sigma_1,\sigma_2} \sigma_1\, \sigma_2 \Big[\hat{P}_{J_1=1,J_2=0}(\sigma_1,\sigma_2) + \hat{P}_{J_1=0,J_2=1}(\sigma_1,\sigma_2)\Big] \qquad (18)$$

and we can compute the average system's energy solving simultaneously the Eq. (16) and Eq. (12), and then applying them into Eq. (18). The reader should notice that, when $J_{ij} = \{0, 1\}$, the energy $\hat{e}$ is directly proportional to the correlation $\hat{c}$ (more explicitly, $\hat{e} = -\lambda\hat{c}$). However, this definition will be of use when the couplings J_{ij} are not binary, as will happen for the Sherrington-Kirkpatrick model in the next section.

In Fig. 1, we compare the results of the average equations for $\lambda = 3$ with Monte Carlo simulations in graphs consisting of $N = 1000$ nodes that were randomly generated using the distribution in Eq. (6). Our average equations are a good description for the system's magnetization and energy for all times and temperatures that were tested. Both simulations and average case predictions display the well-known transition between ferromagnetic and paramagnetic steady states. For $T < 1.8$, the magnetization decays to zero in a short time, while for $T > 1.8$ the system remains magnetized for all times. Since our equations for the magnetization recover the exact result in Ref. [8], we know that all possible discrepancies must come from the finite size effects in the statistics of Monte Carlo simulations.

Through the Eq. (12) we have connected the dynamic cavity method with a known result from a 1987 article [8]. However, in that article, it was assumed that the system's size N was big enough to satisfy $\lambda^m \ll N^{1/2}$, for a system that gets m sequential updates from its starting position.

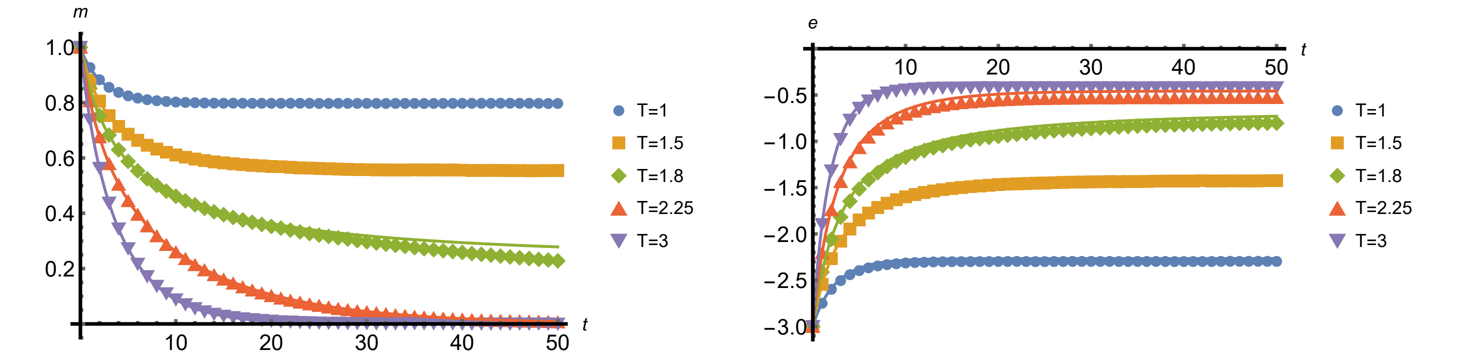

Figure 1. Comparison between the average equations (continuous lines) and Monte Carlo's results (dots) in the fully-asymmetric ferromagnet with $N = 1000$ and $\lambda = 3$. All the calculations were done for a system initially fully magnetized in contact with a heat bath at a given temperature T. Dots are the average for $s = 300$ different graphs. For each one, $n = 10000$ Monte Carlo's histories were averaged.

A common algorithm like Monte Carlo makes typically $O(10N)$ updates, which being $\lambda \geq 2$ implies that the system's size should be big to sustain the former hypothesis.

Our equations do not have such a problem. They have the advantage of being just the average of single-instance equations. This means that we could also reproduce the temporal evolution of a finite system's magnetization and energy. It should also be noted that the energy appearing in Eq. (18) is directly related to the average correlation between the connected variables.

IV. FULLY-ASYMMETRIC SHERRINGTON-KIRKPATRICK

The Sherrington-Kirkpatrick model is a theoretical framework used to describe spin glass systems, introduced in Ref. [12]. It is an Ising model with long-distance interactions where the couplings between spins can be ferromagnetic or antiferromagnetic and they are randomly distributed.

The couplings are drawn from this distribution:

$$Q(J_{ki}) = \sqrt{\frac{N}{2\pi J^2}}\, \exp\{-\frac{N}{2J^2}(J_{ki} - J_0/N)^2\} \tag{19}$$

Unlike the usual Sherrington-Kirkpatrick model, the choice of J_{kl} is independent of J_{lk}. This is the reason for calling it fully asymmetric Sherrington-Kirkpatrick. Notice that, in contrast with the fully asymmetric ferromagnet defined over sparse graphs, the fully-asymmetric Sherrington-Kirkpatrick is defined over fully connected graphs.

IV.1. Magnetization

The local cavity magnetizations $m_j(\sigma_i) = \sum_{\sigma_j} \sigma_j p(\sigma_j \mid \sigma_i)$ follow the following equation:

$$\frac{dm_j(\sigma_i)}{dt} = -\alpha\, m_j(\sigma_i) + \alpha \sum_{\sigma'_j} p(\sigma'_j|\sigma_i) \times \sum_{\sigma_{\partial j\setminus i}} \tanh(\beta \sum_{k\in\partial j} J_{kj}\sigma_k) \prod_{k\in\partial j\setminus i} p(\sigma_k|\sigma'_j) \tag{20}$$

where ∂j has all the nodes in the system, except for σ_j.

Due to the asymmetry of the couplings, the sum $\sum_l J_{lk}J_{kl}$ is of the order $O(1/N)$ and the Onsaguer reaction term is not present. We can then formally establish that the probabilities $p(\sigma_k \mid \sigma'_j)$ will not depend on the couplings J_{kl}, but on J_{lk} with $l \in \partial k$.

Averaging Eq. (20) over the disorder:

$$\frac{d\hat{m}(\sigma)}{dt} = -\alpha\, \hat{m}(\sigma) + \alpha \sum_{\sigma'} \int \Big[\prod_{k=1}^{N-2} dJ_k Q(J_k)\Big] \times \int dJ_i Q(J_i) p_{\{J_k\},J_i}(\sigma'|\sigma) \times \sum_{\{\sigma_k\}} \tanh(\beta \sum_k J_k\sigma_k + J_i\sigma) \prod_k \hat{p}(\sigma_k|\sigma') \tag{21}$$

where we have explicitly written the dependency of each $p(\sigma'_j \mid \sigma_i)$ on the system's couplings. we have defined $\hat{p}(\sigma_k \mid \sigma'_j)$ as the average over all J_{lk} from $p_{\{J_{lk}\}}(\sigma_k|\sigma'_j)$.

As can be seen from Eq. (19) and Eq. (21), the contribution related to J_{ij} within the hyperbolic tangent is of order $O(1/N)$. This means that in the thermodynamic limit we can ignore the term $J_i\sigma$. Also, we can set a starting condition such that $\hat{m}(1) = \hat{m}(-1)$, then we will have a unique equation for both and drop the dependency on σ. Then explicitly integrating over J_i and summing over σ':

$$\frac{d\hat{m}}{dt} = -\alpha\, \hat{m} + \alpha \int \Big[\prod_{k=1}^{N-2} dJ_k Q(J_k)\Big] \times \sum_{\langle\sigma_k\rangle} \tanh(\beta \sum_k J_k\sigma_k) \prod_k \hat{p}(\sigma_k) \tag{22}$$

On the other hand, we can similarly obtain an equation for magnetization defined as $M_j(t) = \sum_{\sigma_j} \sigma_j P_j(\sigma_j)$:

$$\frac{d\hat{M}}{dt} = -\alpha\,\hat{M} + \alpha \int \Big[\prod_{k=1}^{N-2} dJ_k Q(J_k)\Big] \times \\ \times \sum_{\sigma_k} \tanh(\beta \sum_k J_k\sigma_k) \prod_k \hat{p}(\sigma_k) \tag{23}$$

If we set a starting condition such that $\hat{m}(0) = \hat{M}(0)$, we will have $\hat{m}(t) = \hat{M}(t)$ for all $t > 0$, and just one equation:

$$\frac{d\hat{m}}{dt} = -\alpha\,\hat{m} + \alpha \int \Big[\prod_{k=1}^{N-2} dJ_k Q(J_k)\Big] \times \\ \times \sum_{\{\sigma_k\}} \tanh(\beta \sum_k J_k\sigma_k) \prod_k \hat{p}(\sigma_k) \tag{24}$$

Then, we need to compute the Gaussian integral in the second term on the right side:

$$I(\hat{m}, \beta, J_0, J, \eta) = \int \Big[\prod_{k=1}^{N-2} dJ_k Q(J_k)\Big] \times \\ \times \sum_{\{\sigma_k\}} \tanh\big[\beta(\sum_k J_k\sigma_k + \eta)\big] \prod_k \hat{p}(\sigma_k) \tag{25}$$

As all variables J_k follow a Gaussian distribution, the variable $\zeta = \sum_k J_k\sigma_k$ will also follow a Gaussian distribution. Then defining $Dy \equiv e^{-y^2/2}/\sqrt{2\pi}$ where $y = (\zeta - \langle\zeta\rangle)/J^2$ we get:

$$I(\hat{m}, \beta, J_0, J, \eta) = \int Dy \sum_{\{\sigma_k\}} \Big[\prod_k \hat{p}(\sigma_k)\Big] \tanh\big[\beta(\frac{J_0}{N}\sum_k \sigma_k + J\,y + \eta)\big] \tag{26}$$

For $N \gg 1$, the variable $h = \frac{1}{N}\sum_k \sigma_k$ is a sum of a great number of independent and identically distributed variables, also distributed as a Gaussian. Therefore, at the thermodynamic limit we get $I(\hat{m}, \beta, J_0, J, \eta)$:

$$I(\hat{m}, \beta, J_0, J, \eta) = \int Dy \tanh\big[\beta(J_0\,\hat{m} + J\,y + \eta)\big] \tag{27}$$

The equation for system's magnetization is then:

$$\frac{d\hat{m}}{dt} = -\alpha\,\hat{m} + \alpha I(\hat{m}, \beta, J_0, J, 0) \tag{28}$$

which reproduces the analytical results presented in [9].

IV.2. Energy

We will also obtain an equation for the energy of the system as we did in Eq. (17):

$$\hat{e}(t) = -\frac{N}{2}\int dJ_1 dJ_2 Q(J_1, J_2)\frac{(J_1 + J_2)}{2} \times \sum_{\sigma_1,\sigma_2} \sigma_1\,\sigma_2\,\hat{P}_{J_1,J_2}(\sigma_1, \sigma_2) \tag{29}$$

As last time, this is not the usual energy; it is more like a measure of the intensity of spin interactions. Starting from an analogous equation to Eq. (16):

$$\begin{aligned}\frac{d\hat{P}_{J_1,J_2}(\sigma_1, \sigma_2)}{dt} = &-\frac{\alpha}{2}\sum_\sigma (\sigma\sigma_1\hat{P}_{J_1,J_2}(\sigma, \sigma_2) + \sigma\sigma_2\hat{P}_{J_1,J_2}(\sigma_1, \sigma)) \\ &+ \frac{\alpha\sigma_1}{2}\Big[\sum_\sigma \hat{P}_{J_1,J_2}(\sigma, \sigma_2)\Big] I(\hat{m}, \beta, J_0, J, J_2\sigma_2) \\ &+ \frac{\alpha\sigma_2}{2}\Big[\sum_\sigma \hat{P}_{J_1,J_2}(\sigma_1, \sigma)\Big] I(\hat{m}, \beta, J_0, J, J_1\sigma_1)\end{aligned} \tag{30}$$

We can then apply the operator:
$-N/2 \int dJ_1 dJ_2 Q(J_1, J_2)(J_1 + J_2)/2 \sum_{\sigma_1}\sum_{\sigma_2} \sigma_1\sigma_2[\cdot]$ and get:

$$\begin{aligned}\frac{d\hat{e}(t)}{dt} = &-2\alpha\hat{e}(t) - \frac{\alpha N}{2}\sum_\sigma \sigma\Big[\int dJ_1 Q(J_1)\,J_1\,\hat{P}_{J_1}(\sigma)\Big] \\ &\times \int dJ_2 Q(J_2) I(\hat{m}, \beta, J_0, J, J_2\sigma) \\ &- \frac{\alpha N}{2}\sum_\sigma \sigma\Big[\int dJ_1 Q(J_1)\,\hat{P}_{J_1}(\sigma)\Big] \\ &\times \int dJ_2 Q(J_2)\,J_2\,I(\hat{m}, \beta, J_0, J, J_2\sigma)\end{aligned} \tag{31}$$

We know that in the thermodynamic limit:

$$\int dJ_2 Q(J_2) I(\hat{m}, \beta, J_0, J, J_2\sigma) = I(\hat{m}, \beta, J_0, J, 0) \tag{32}$$

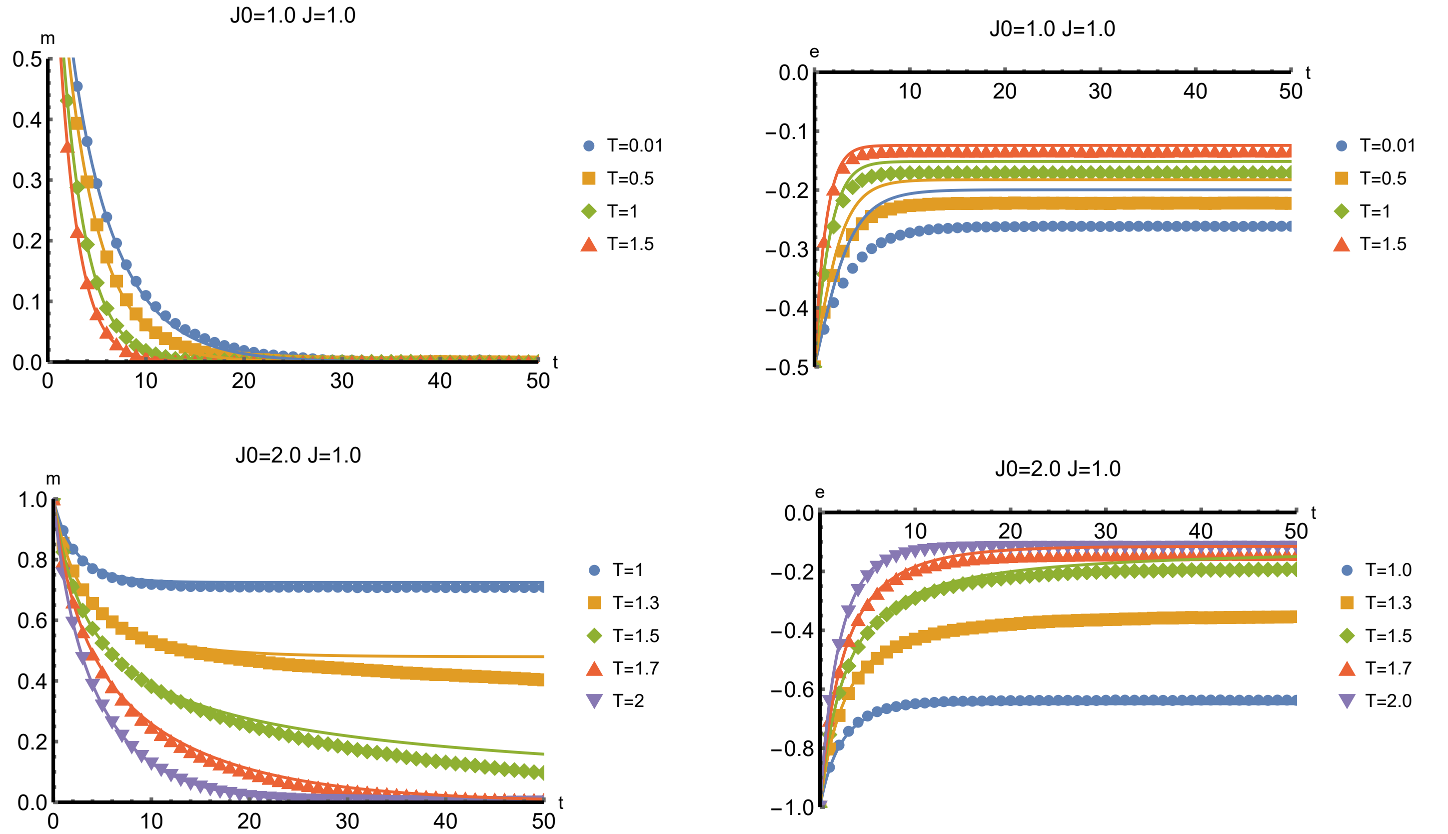

Figure 2. Comparison between average equations (continuous lines) and Monte Carlo's results (dots) in the fully-asymmetric Sherrington-Kirkpatrick. All calculations were done for an initially fully magnetized system in contact with a heat bath at a given temperature T. For each one graph, $n = 100$ Monte Carlo's histories were averaged. Panels (a) and (b): System size is $N = 500$. Points are averages taken over $s = 100$ graphs. Panels (c) and (d): System size is $N = 100$. Points are averages taken over $s = 1000$ graphs.

This leaves us with the task of solving the following integrals:

$$\begin{aligned} N\langle J_1 \hat{m}_{J_1}\rangle \; &\equiv N\int dJ_1 Q(J_1)\, J_1\, \hat{P}_{J_1}(\sigma) \\ N\langle J_2\, I(J_2\sigma)\rangle \; &\equiv N\int dJ_2 Q(J_2)\, J_2\, I(\hat{m},\beta,J_0,J,J_2\sigma) \end{aligned} \tag{33}$$

where $\hat{m}_{J_1} = \sum_\sigma \sigma \hat{P}_{J_1}(\sigma)$.

In all of the following derivations, we will use the fact that the integrals are uniformly convergent. Now let's integrate Eq. (33) by parts with:

$$u = \int Dy\, \tanh\left[\beta(J_0\hat{m} + Jy + J_2\sigma)\right] \tag{34}$$

$$dv = dJ_2\sqrt{\frac{N}{2\pi J^2}}\exp\left\{-\frac{N}{2J^2}(J_2 - \frac{J_0}{N})^2\right\} J_2 \tag{35}$$

Then, we have:

$$N\langle J_2\, I(J_2\sigma)\rangle = N(uv\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} vdu) \tag{36}$$

Using $\text{lím}_{x\to\pm\infty}\,\text{erf}(x) = \text{lím}_{x\to\pm\infty}\tanh(x) = \pm 1$:

$$uv\big|_{-\infty}^{\infty} = \frac{J_0\sigma}{2N}\int Dy - \frac{J_0\sigma}{2N}\int Dy = 0 \tag{37}$$

In the other side:

$$\int_{-\infty}^{\infty} vdu = I_1 + I_2 \tag{38}$$

where we have:

$$\begin{aligned} I_1 = \beta\sigma\int_{-\infty}^{\infty} dJ_2\left(-\frac{J^2}{N}\sqrt{\frac{N}{2\pi J^2}}\exp\left\{-\frac{N}{2J^2}(J_2 - \frac{J_0}{N})^2\right\}\right) \\ \times\int Dy\, \cosh^{-2}\left[\beta(J_0\hat{m} + Jy + J_2\sigma)\right] \end{aligned} \tag{39a}$$

$$\begin{aligned} I_2 = \beta\sigma\int_{-\infty}^{\infty} dJ_2\frac{J_0}{2N}\text{erf}\left[\sqrt{\frac{N}{2J^2}}(J_2 - \frac{J_0}{N})\right] \\ \times\int Dy\, \cosh^{-2}\left[\beta(J_0\hat{m} + Jy + J_2\sigma)\right] \end{aligned} \tag{39b}$$

At Eq. (39a) we have the factor $\exp\left\{-\frac{N}{2J^2}(J_2 - \frac{J_0}{N})^2\right\}$, which when N is big, the gaussian will be localized around $J_2 = \frac{J_0}{N}$. Therefore is safe to make the substitution $J_2 = \frac{J_0}{N}$ inside the hyperbolic cosine and then integrating over J_2:

$$I_1 \approx -\frac{\beta\sigma J^2}{N}\int Dy\,\cosh^{-2}\left[\beta(J_0\hat{m}+Jy+\frac{J_0}{N}\sigma)\right] \tag{40}$$

At Eq. (39b), we have the factor $\operatorname{erf}\left[\sqrt{\frac{N}{2J^2}}(J_2-\frac{J_0}{N})\right]$. When N is big enough, $\operatorname{erf}\left[\sqrt{\frac{N}{2J^2}}(J_2-\frac{J_0}{N})\right] \approx sgn(J_2-\frac{J_0}{N})$.

Then exchanging the integral signs and integrating:

$$I_2 \approx -\frac{J_0}{N}\int Dy\,\tanh\left[\beta(J_0\hat{m}+Jy+\frac{J_0}{N}\sigma)\right] \tag{41}$$

Putting it all together and ignoring the $\frac{J_0}{N}\sigma$ inside the hyperbolic functions because they vanish in the thermodynamic limit, we get finally:

$$\begin{aligned} N\langle J_2\, I(J_2\sigma)\rangle = J_0\int Dy\,\tanh\left[\beta(J_0\hat{m}+Jy)\right] \\ +\beta\sigma J^2\int Dy\,\cosh^{-2}\left[\beta(J_0\hat{m}+Jy)\right] \end{aligned} \tag{42}$$

To get an equation for $N\langle J_1\hat{m}_{J_1}\rangle$ we just multiply by J_1 the differential equation for the variable $\hat{m}_{J_1}$, and then integrating the result with weight $Q(J_1)$:

$$\begin{aligned} N\frac{d\langle J_1\hat{m}_{J_1}\rangle}{dt} = -\alpha N\langle J_1\hat{m}_{J_1}\rangle + \alpha J_0\int Dy\tanh\left[\beta(J_0\,\hat{m}\right. \\ \left.+J\,y)\right] + \alpha\hat{m}\beta J^2\int Dy\cosh^{-2}\left[\beta(J_0\,\hat{m}+J\,y)\right] \end{aligned} \tag{43}$$

where there was done an integration by parts to analogously get the second and third terms.

We can solve numerically the system of equations formed by Eq. (31), Eq. (32), Eq. (42) and Eq. (43) with starting conditions $\hat{m}(0)=m_0$, $\hat{e}(0)=-\frac{J_0m_0^2}{2}$ and $N\langle J_1\hat{m}_{J_1}\rangle(0)=J_0m_0$.

In Figs. 2a and 2b, the results from the average equations are compared with Monte Carlo simulations, for $J_0=1, J=1$ and $N=500$. The system's magnetization is well described by our theory, which is to be expected since we again recover known results from the literature that are exact. In the same system, our equations predict steady state energies a bit higher than the ones obtained in the simulations. When $J_0=J=1$, there is no ferromagnetic region and the dynamics always go to a non-magnetized steady state. This does not mean the dynamics is trivial, since correlations do emerge when the temperature is lowered. This mechanism, possibly associated with a glassy dynamics, is not captured in Fig. 2a, where the theory is close to the simulations for high temperatures but fails for low temperatures.

However, we can show that the equations work better in systems with stronger ferromagnetic interactions. In Figs. 2c and Figs. 2d, we can see that for $J_0=2, J=1$ and $N=100$, our theory describes adequately the system's magnetization and the energy as well. Here, we do have a ferromagnetic-paramagnetic transition and the emergence of correlations for low temperatures is due to ferromagnetic interactions. This is, instead, well captured by our equations.

V. CONCLUSIONS

We developed a method to get average case versions of the cavity master equation [1] in asymmetric models. The ideas and the methods can be easily extended to other models.

Our method recovers exact results already known in the literature for the magnetizations of the fully-asymmetric ferromagnet and Sherrington-Kirkpatrick models. Furthermore, we obtain new equations for the energy of both models that reproduce the simulations in most cases.

For specific parameters our equations predict steady-states energies a bit higher than those of the simulations in the fully-asymmetric Sherrington-Kirkpatrick. This is associated to the emergence of non-trivial correlations in the dynamics without magnetization, possibly due to the presence of a glassy dynamics. We plan to analyze the discrepancies observed in this particular case in the future.

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