THE TWO-STATE DIFFUSION PROBLEM WITH RESETTING

REAJUSTE ESTOCÁSTICO PARA LA DIFUSIÓN EN DOS ESTADOS

J. J. Díaz-Pérez[a†] and R. Mulet[b]

a) Facultad de Matemática y Computación, Universidad de La Habana. jjdpbeat44@gmail.com[†]
b) Facultad de Física, Universidad de La Habana, Cuba. roberto.mulet@gmail.com
† corresponding author

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The concept of stochastic resetting emerged in 2011, introducing a simple modification to [15] the one dimensional Random Walk that leads to a stationary distribution and where the first-passage time can be easily optimized. In this context, we will investigate what happens when modeling these phenomena in a framework where a Brownian particle alternates between different states. We will derive the Fokker-Planck equations of the model and analyze the behavior of the probability density of the particle positions for long times and solve the corresponding first-passage time problem. Contrasting with the two-state Brownian motion, we will observe how resetting ensures a stationary behavior in the long run leading to a finite expectation for the first-passage problem. Finally, we will demonstrate how the existence of this expectation guarantees minimum values by varying the resetting or state-change parameters.

La idea del reajuste estocástico aparece en 2011 abriendo la posibilidad de interpretar un nuevo grupo de fenómenos, y sobre todo llamando la atención sobre las propiedades favorables de este tipo de modelos. Con este sentido estudiaremos qué sucede al modelar este tipo de fenómenos en un espacio de alternancia de los dos estados en que se mueve una partícula browniana. Deduciremos las ecuaciones de Fokker-Planck del modelo y a partir de ellas trabajaremos el comportamiento de la partícula para tiempos largos y el problema del primer cruce. En comparación con el movimiento browniano clásico en dos estados podremos ver cómo el reajuste nos garantiza un comportamiento estacionario para tiempos largos, y por tanto una esperanza finita para el problema del primer cruce. Por último, veremos como la existencia de la esperanza nos garantiza valores mínimos de esta al realizar variaciones sobre los parámetros de reajuste o cambio de estados.

PACS: Stochastic resetting (restablecimiento estocástico); double diffusivity (doble difusividad); first-passage time (tiempo de primer paso); Fokker-Planck Equation (ecuación de Fokker-Planck); brownian motion (movimiento browniano).

I. INTRODUCTION

The Brownian motion (BM) is one of the most studied stochastic processes of nature. It has been extensively used to model the molecular movement in a fluid [1], financial trends [2], animal foraging behavior [3], optimal search algorithms [4], and others. Among the various variations of the BM, in reference [5] the authors add to the Brownian motion a stochastic resetting process that follows a Poisson distribution. While the standard Browian motion does not reach a stationary distribution, introducing a stochastic resetting provides to the process a stationary character and a finite expectation for the first-passage problem. These authors have also shown that there exists a value of resetting rate r that minizes this expectation [5].

The idea of stochastic resetting has gained special relevance in the last decade due to its ability to describe a variety of processes in different disciplines. In this way, its incorporation can be observed in various variations of Brownian motion: drifted BM [6], scaled BM [7], and fractional BM [8]; as well as in different studies on animal behavior [9] and genome analysis [10].

To exemplify our problem, we can consider a miner searching for gold in a jungle as a high uncertainty environment. Let us assume that the miner motion is characterized by a diffusion constant, and that the value of this diffusion varies depending on whether it rains (low diffusion) or it is sunny. This corresponds to a Brownian motion in a two state problem.

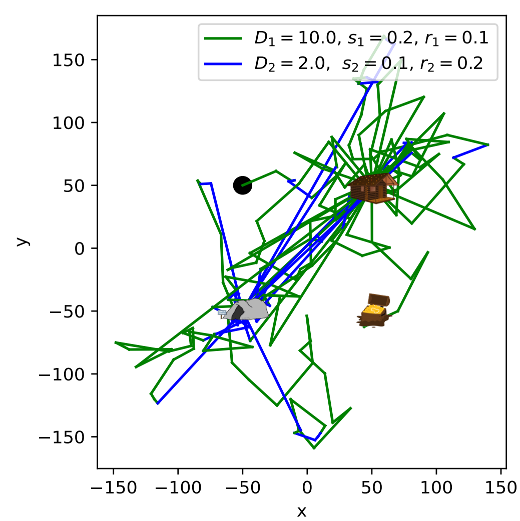

Figure 1. Trajectories of a miner searching for gold ina jungle. (-50,50) is the starting point of the miner. The house and the cave are the points to which he resets depending on whether the weather is sunny or rainy, respectively. The journey ends in this case when he finds the gold. The legend shows the diffusion constants in each state and the rates for resetting and state change.

The two-state problem has a significant body of work in the literature [11–13]. However, similar to the standard Brownian motion, this process will never reach a steady state, and the expectation of the first-passage time is also divergent [14]. Here, we will ask questions such as, what will happen if the miner returns to a specific fixed point with a certain probability based on the weather conditions? How will the miner's search behavior evolve over long times? Will the expectation of the miner finding gold be finite? Is it significant that the resetting rate depends on the weather state? (see Figure 1).

We focus on the general one-dimensional modeling of Brownian motion with resetting in two states, addressing it in its most general form, which involves problems with two states and two resets to different positions [15]. Using the Fokker-Planck equation, we evaluate the system's behavior over long times. Then, by leveraging the stationary characteristics of the process, we explore the convergence of the first-passage expectation and the existence of a non-trivial minimum for a single resetting point. The primary reason for considering a single resetting point in the first-passage problem is to avoid added complexity in this initial approach. We chose to work with the relationship between diffusions rather than the relationship involving diffusion coefficients and reset points. Finally, we examine the behavior of this minimum across various parameter sets.

II. THE MODEL AND FOKKER-PLANCK EQUATION

Before obtaining the equations that will describe our Two-State Stochastic Resetting Brownian motion (BMR2S), let's frame its definition properly for one dimension. The problem to be addressed is that of a particle that starts at a point x_0 in state i with probability q_i, and for each moment $t \geqslant 0$, if it is in state i, at instant dt it can

- transit from state i to state $j \neq i$ with rate $s_j\mathrm{d}t$ $(i, j = 1, 2)$,
- move with BM of diffusion D_i,
- reset its position to a fixed position x_i at a rate of $r_i\mathrm{d}t$.

Let $P_i(x, t)$ denote the density function describing the probability of the particle in a BMR2S motion being at position x in state i at time t. Additionally, we will take $\phi^i_{\Delta t}(\Delta x)$ as the density function of the Brownian motion with diffusion D_i (i.e. a Gaussian function over Δx with distribution $N(0, 2D\Delta t)$), and $\mathcal{P}_i(t)$ as the probability of the particle being in state i at time t. The description of our BMR2S allows us to consider, for $\Delta t << 1$, the equations

$$\begin{aligned} P_i(x, t + \Delta t) &= \int_{-\infty}^{\infty} \phi^i_{\Delta t}(\Delta x) P_i(x - \Delta x, t)\mathrm{d}(\Delta x) \\ &+ s_i \Delta t P_j(x, t) + r_i \Delta t \mathcal{P}_i(t)\delta(x - x_i) \end{aligned} \tag{1}$$

for $i \neq j \in \{1, 2\}$ and $\delta(y)$ the Dirac delta function.

Regarding the functions $\mathcal{P}_i(t)$, it is sufficient to note that $\mathcal{P}_1(t) + \mathcal{P}_2(t) = 1$, and $\mathcal{P}_1(t)$ is given by

$$\mathcal{P}_1(t) = q_1 \exp[-t(s_1 + s_2)] + \frac{s_1 - s_1 \exp[-t(s_1 + s_2)]}{s_1 + s_2} \tag{2}$$

where q_1 represents the probability of starting in state 1. For simplicity, we set $q_1 = \frac{s_1}{s_1+s_2}$, ensuring stationary behavior over time, i.e., $\mathcal{P}_i = \frac{s_i}{s_1+s_2}$.

The three terms at (1) describe, in a time interval Δt, the possible arrivals at position x at time $t + \Delta t$. These are: the probability of moving in a BM with diffusion D_i, the probability of transitioning to state i from state j, and the probability of a resetting occurrence to position x_i. It is important to note that as $\Delta t \to 0$,

$$\int_{-\infty}^{\infty} \phi^i_{\Delta t}(\Delta x)\mathrm{d}(\Delta x) + r_i \Delta t + s_j \Delta t \approx 1. \tag{3}$$

Proceeding similarly to the literature (see ref. [8]) on (1) and using (3), we obtain that the Fokker-Planck equations are, for $i \in \{1, 2\}$:

$$\begin{aligned} \frac{\partial P_i(x, t)}{\partial t} &= D_i \frac{\partial^2 P_i(x, t)}{\partial x^2} - (r_i + s_j) P_i(x, t) \\ &+ s_i P_j(x, t) + r_i \mathcal{P}_i(t)\delta(x - x_i). \end{aligned} \tag{4}$$

with $j \neq i \in \{1, 2\}$. Once we have the Fokker-Planck equations derived, we can now address the behavior of $P_i(x, t)$ for long times and the first-passage time problem.

III. LONG-TIME DESCRIPTION

The probability density equation of the process as a function of position and time is given by $P(x, t) = P_1(x, t) + P_2(x, t)$. Due to the uniqueness of the solution [12] of our Fokker-Planck system (4), it is sufficient to find just one solution to the problem. We then perform both a Laplace and Fourier transforms to transform the partial differential equation in an alegbraic expression whose inverse determines the probability density of the problem for all times. However starting from the well-known limit [16] it is straighforward to show that, if we denote as $\mathcal{L}$ the Laplace transform,

$$\lim_{z \to 0} z\mathcal{L}[f(t)](z) = \lim_{t \to \infty} f(t), \tag{5}$$

that defines the stationary behavior of our function and avoids the complex calculation of the inverse of the transforms.

Computing the previous limit, we find that when taking $\Lambda^{\pm}_i = \sqrt{\lambda_+} e^{-\sqrt{\lambda_{\pm}}|x - x_i|} - \sqrt{\lambda_-} e^{-\sqrt{\lambda_{\mp}}|x - x_i|}$, with $\lambda_{\pm} = \frac{(s_1+r_2)}{2D_2} + \frac{(s_2+r_1)}{2D_1} \pm \sqrt{\left[\frac{(s_1+r_2)}{2D_2} - \frac{(s_2+r_1)}{2D_1}\right]^2 + \frac{s_1 s_2}{D_1 D_2}}$, our stationary distribution P_{ss} will be

$$\begin{aligned} P_{ss}(x) &= \frac{r_1 s_1 \left(D_2 \sqrt{\lambda_+ \lambda_-} \Lambda^+_1 + (s_1 + s_2 + r_2)\Lambda^-_1\right)}{2D_1 D_2 (s_1 + s_2) \sqrt{\lambda_+ \lambda_-}(\lambda_+ - \lambda_-)} \\ &+ \frac{r_2 s_2 \left(D_1 \sqrt{\lambda_+ \lambda_-} \Lambda^+_2 + (s_1 + s_2 + r_1)\Lambda^-_2\right)}{2D_1 D_2 (s_1 + s_2) \sqrt{\lambda_+ \lambda_-}(\lambda_+ - \lambda_-)}. \end{aligned} \tag{6}$$

Unlike the case of two states without stochastic resetting, in our scenario, we will have a well-defined stationary state, the behavior of which can be observed in Fig. 2.

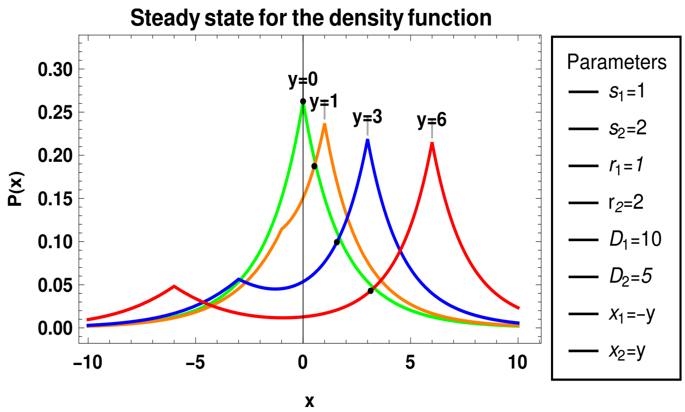

Figure 2. Behaviour of the density function (6) for a given set of parameters over very large times, i.e. as $t \to \infty$. The figure illustrates the relationships between the parameters and how they influence the behavior of the probability density function. In this case, $s_2 > s_1$, $r_2 > r_1$, and $D_1 > D_2$ contribute to a higher probability concentration around x_2, meaning that the particle is more likely to stay near x_2 for long times. The black dots represent the expected value for each probability density function.

If we quickly review equation (6), we observe that there are two weights, one for each x_i. By adjusting the parameters D_i, r_i, and s_i, we can see how these weights vary almost proportionally (inversely for D_i and directly for r_i and s_i). In summary, if we look at *probability concentration* around x_i as the probability that the particle is within some vicinity of x_i, we find that:

- $D_i > D_j$ results in a lower probability concentration at x_i with respect to x_j,
- $r_i > r_j$ leads to a higher probability concentration at x_i with respect to x_j,
- $s_i > s_j$ leads to a higher probability concentration at x_i with respect to x_j,
- If $x_1 = x_2$, $r_1 = r_2$, and $D_1 = D_2$, the behavior is the same as for a single state with resetting.

IV. FIRST-PASSAGE TIME PROBLEM

The mean first-passage time of a stochastic process is defined in the following way: given a stochastic process $X(t)$ with initial condition $x_0 > 0$, what is the the expectation of the random variable representing the first passage time to $x = 0$? In short, the expectation of the random variable $t_f = \text{ínf}_t\{X(t) \leqslant 0\}$).

To compute this quantity we make use of the well-known Chapman-Kolmogorov equation for Markov stochastic processes,

$$P(0,t|x_0,0) = \int_0^t \mathbb{F}(x_0,\tau)P(0,t|0,\tau)d\tau, \tag{7}$$

where P and $\mathbb{F}$ represent the probability densities of the process under consideration and the first-passage, respectively. To calculate the expectation of the first-passage $T(x_0)$, we exploit a formula previously derived from the properties of the Laplace Transform and the survival probability function [8,14],

$$T(x_0) = \lim_{z\to 0} \frac{1-\hat{\mathbb{F}}(x_0,z)}{z}. \tag{8}$$

In other words, given the equation (7), we only need the Laplace transform of $\mathbb{F}$ to calculate T. This is not difficult, because of the Laplace Transform of a convolution of two functions is the product of the Laplace transform of these functions.

The limit of equation (8) is confirmed to exist if at least one of the inequalities $r_1s_1 \neq 0$ or $r_2s_2 \neq 0$ holds. This means that there must be at least one state with non-zero stochastic resetting and a non-zero rate of transition to it.

Next, we derive the equation for the expectation T and simplify it by introducing the substitutions $\alpha = \frac{D_1}{D_2}$ and $\theta = \frac{x_0}{\sqrt{D_2}}$. Here, α denotes the ratio of diffusions in both states, and θ represents the relationship between the resetting position and the displacement. Consequently, our expression for the expectation (8) can be simplified to $T = \frac{\Psi}{\Phi}$, where

$$\begin{aligned}
\Psi &= \sqrt{\mu_+}\left(1-e^{-\sqrt{\mu_-}\theta}\right)[\beta-(s_1+s_2\alpha)\mu_-] \\
&- \sqrt{\mu_-}\left(1-e^{-\sqrt{\mu_+}\theta}\right)[\beta-(s_1+s_2\alpha)\mu_+] \\
\Phi &= \sqrt{\mu_+}e^{-\sqrt{\mu_-}\theta}\left[\gamma-(r_1s_1+\alpha r_2s_2)\mu_-\right] \\
&- \sqrt{\mu_-}e^{-\sqrt{\mu_+}\theta}\left[\gamma-(r_1s_1+\alpha r_2s_2)\mu_+\right]
\end{aligned} \tag{9}$$

with $\mu_\pm := \frac{\alpha(s_1+r_2)+(s_2+r_1)\pm\sqrt{[\alpha(s_1+r_2)-(s_2+r_1)]^2+4\alpha s_1s_2}}{2\alpha}$ $\beta = (s_1+s_2)^2 + s_1r_2 + s_2r_1$ and $\gamma = (s_1+s_2)(r_1s_1+r_2s_2+r_1r_2)$.

With this expression, we then ask, in a similar way as in ref. [5], whether the expectation of the first-passage time can be minimized by adjusting the resetting parameters r_1 and r_2. To address this, we numerically compute the values of r_1 and r_2 that minimize T for specific parameter sets. As mentioned at the beginning, we assume in this case that $x_0 = x_1 = x_2$.

Let's denote this minimum of T as T_m. Subsequently, we compare this with the outcome of optimizing T considering a single parameter $r = r_1 = r_2$. In this case, the miner returns to its original position at a rate that does not depend on its state. We denote this minium as T_r. These two values are compared with the extreme cases, i.e., with the prediction obtained in ref. [5] for one state. We denote this time as: $T = T_e$ and is obtained when $r_i = r_i^* = \frac{c^2D_i}{x_0^2}$, where $c \approx 1.5441$. It is not difficult to verify that these r_i^* values yield the minimum of T when our two-state process can be simplified to a single diffusion process—either due to equivalent diffusions in both states (i.e., $D_1 = D_2$) or when the mean time in one of the two states is zero (i.e., $s_1s_2 = 0$).

Let us first examine how T_e depends on α, θ, s_1, and s_2. If we fix θ and increase the value of α, it is equivalent to enhancing the diffusion in one of the states. In this scenario, as diffusion increases, the expectation time decreases, indicating an increased probability of an earlier occurrence of the first-passage (see Figure 3).

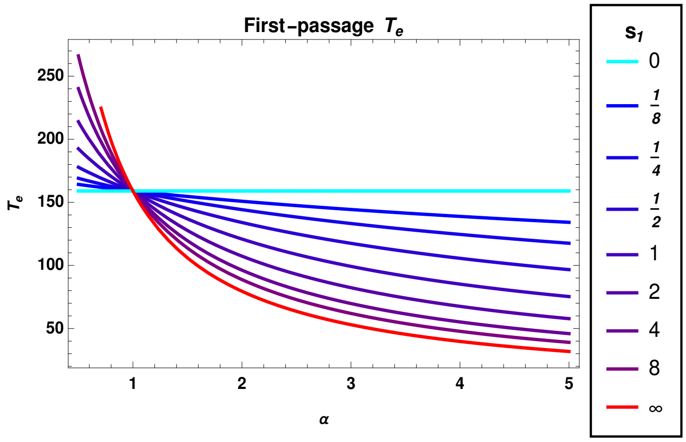

Figure 3. Dependence of T_e on α for different values of s_1. In all cases, $\theta = 10$ and $s_2 = 2$ are used. It can be checked that when $\alpha = 1$, the resetting point is unique, and the minimum $T_m = T_e$ remains constant regardless of the values of s_1 and s_2. For the extreme cases $s_1 = 0$ and $s_1 = \infty$, it can be verified that these are equivalent to a scenario in which there is a transition between the two states, but both have equal diffusion. Specifically, $s_1 = 0$ corresponds to the case where both states have diffusion D_2, and $s_1 = \infty$ corresponds to the case where both states have diffusion $D_1 = \alpha D_2$. In both cases, the values of the parameters s_1 and s_2 do not affect the expected value.

On the other hand, for $\alpha > 1$, we observe that the expectation T_e will be bounded below by the case where $\frac{s_1}{s_2} \to \infty$, representing a situation in which the particle primarily moves in the state with large diffusion. It can be verified that the global minimum of T, when optimizing over (s_1, s_2, r_1, r_2), is T_e as $\frac{s_1}{s_2} \to \infty$. The alternative scenario occurs when $\frac{s_1}{s_2} \to 0$, which corresponds to the particle spending most of its time in the state with low diffusion. Figure 3 illustrates how the values of $\frac{s_1}{s_1+s_2}$ and $\frac{s_2}{s_1+s_2}$ function as a kind of weighted average between the two extremes, $\frac{s_1}{s_2} \to \infty$ and $\frac{s_1}{s_2} \to 0$. Overall, this reinforces the notion that a larger *average diffusion* leads to a reduction in the expectation value of the first-passage time.

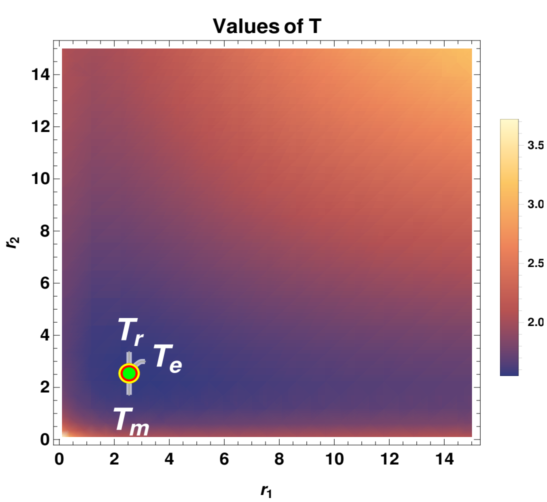

Figure 4. Values of $T(r_1, r_2)$ given the values of $(s_1, s_2, \alpha, \theta) = (1, 2, 1, 1)$. It can be observed how the values of T_m, T_r, and T_e coincide, starting from the condition $\alpha = 1$. As shown in Figure 3, it can also be analytically verified that the values of s_1 and s_2 have no impact on the outcome.

In terms of the miner's logic, this would imply, as a first approximation, that the higher the ratio of sunny to rainy days—favoring sunny weather—the probability of finding the gold mine increases with the number of sunny days and reaches its maximum when it never rains. Let's now see what happens optimizing with respect to the two resetting parameters (r_1, r_2). We first look to the case $\alpha = 1$ (i.e. $D_1 = D_2$). In this scenario, setting $r = r_1 = r_2$ in the expression of T (9), it can be shown analytically that the optimal solution is achieved when $r = r_1^* = r_2^*$. Under this condition, $T_r = T_e$. Moreover, we can show that optimizing with respect to both parameters, r_1 and r_2, is equivalent to fix them to $r_1 = r_1^*$ and $r_2 = r_2^*$. Therefore, for $D_1 = D_2$ and a single resetting point, the values of T_r, T_m, and T_e coincide (see Figure 4).

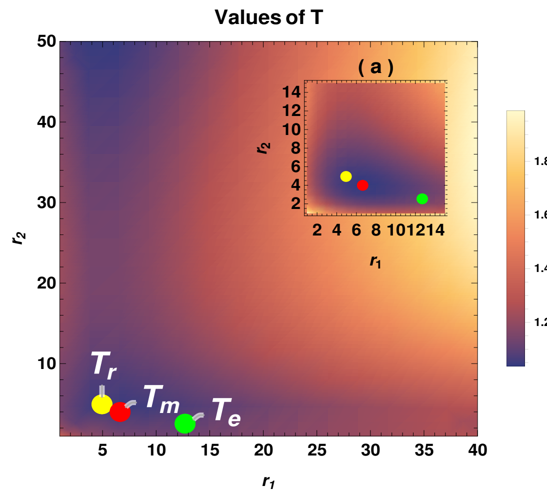

Figure 5. Values of $T(r_1, r_2)$ analyzed for $(s_1, s_2, \alpha, \theta) = (1, 2, 5, 1)$. In (a), the graph is enlarged around which T_m, T_r, and T_e are found. It is important to note that in this case, T_m as a minimum is taken locally, that is, it is numerically calculated taking T_r as the starting point, so T_m is not necessarily a minimum over $(r_1, r_2) \in \mathbb{R}_+^2$.

Now, let's assume without loss of generality, that $\alpha > 1$. Initially, as shown in Figure 5.a, for the parameter values $(s_1, s_2, \alpha, \theta) = (1, 2, 5, 1)$, there exists a local minimum T_m reached when $r_1 \neq r_2$. This local minimum, a candidate for T_m, is already lower than both T_r and T_e (Figure 5). Moreover, it is notable from Figure 5 that beyond a certain threshold of r_2, the values of T starts to decrease, indicating the possibility of another global minimum.

Indeed, numerous numerical simulations show that letting $r_2 \to \infty$ often yields a global minimum. What does this mean? When $r_2 \to \infty$, it effectively removes diffusive movement in state 2 (the state with lower diffusion). Each time the system enters state 2, the particle immediately resets to position x_0 and remains there for an exponential period corresponding to the time spent in state 2. Movement resumes only when transitioning to state 1. In the miner language: *do not go outside if it's raining*.

To deduce the value of r_1 for which T reaches its minimum when $r_2 \to \infty$, let's observe that the transition to state 2 acts as a resetting at a rate of s_2, and the time spent in state 2 serves as a penalty on the first-passage time, as the particle remains at x_0. This scenario becomes equivalent to a stochastic resetting with one diffusion state process, where the resetting parameter is $r_1 + s_2$, and for each trajectory, each transition to s_2 penalizes the time of the first-passage with an exponential random variable of parameter s_1.

Therefore, the minimum would be given by the optimal resetting with respect to the single-state diffusion problem [5], taking into account the value of the state change. That is, when $r_2 \to \infty$, the optimal value for r_1 will be given by

$$r_1 = \text{máx}\left\{0, \frac{c^2 D_1}{x_0^2} - s_2\right\}. \quad (10)$$

The fact that 0 is the best value when $\frac{c^2 D_1}{x_0} - s_2 < 0$ results from the monotonicity of the expectation over the resetting parameter for the single-state problem, as in such a case the parameter s_2 will act as the resetting parameter.

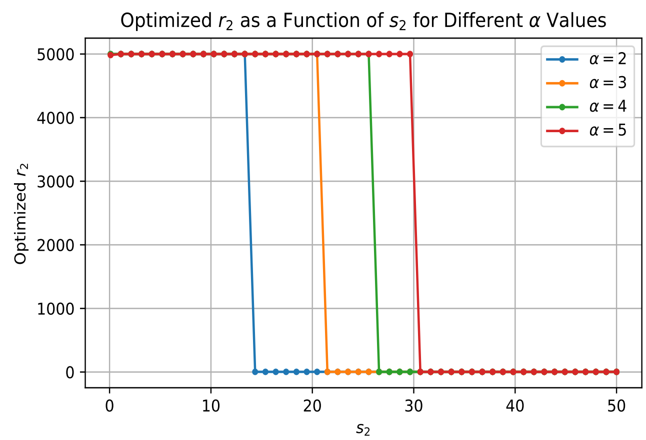

Figure 6. Optimized $\theta = 1$ and a function of $s_1 = 2$ for different $\alpha = 2, 3, 4, 5$ values. The calculations are performed numerically, with 5000 used to represent a sufficiently large value that approximates $r_2 \to \infty$. When r_2 decreases, it reaches a local minimum, as shown in Figure 5, so r_2 is greater than zero but very small compared to 5000 in this case. It can be observed the phase transition of r_2 based on the values of s_2.

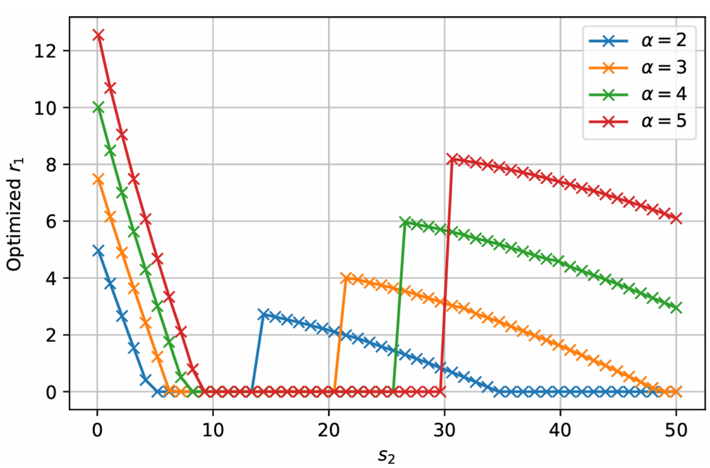

Figure 7. For $\theta = 1$ and $s_1 = 2$, this figure illustrates the behavior of r_1 as a function of s_2 for $\alpha = 2, 3, 4, 5$. Initially, r_1 decreases linearly to zero, consistent with Equation (10). Beyond a critical s_2 point, the local minimum becomes the global minimum, resulting in a jump to a non-zero r_1 value, which then begins to decrease again.

Examining the global minimum between the local candidate and the candidate obtained by letting r_2 tend to infinity, we observe a transition that depends on the relationship between s_1 and s_2. Specifically, if we fix s_1 we find that for low values of s_2 (i.e., less time in state 2), letting r_2 tend to infinity yields the global minimum. However, for high values of s_2 (i.e., when it 'rains' frequently, in terms of the miner's logic), the local minimum actually represents the global minimum (see Figure 6).

Moreover, we see in Figure 7 that the behavior of r_1 aligns with the prediction made in Equation (10), specifically for the values where $r_2 \to \infty$ in Figure 6. In this case, r_1 decreases linearly as a function of s_2 until it reaches zero. However, after the transition point, where r_2 becomes finite in relation to s_2, a local minimum appears for both r_1 and r_2, with $r_1 \neq 0$. Beyond this point, for sufficiently large values of s_2, r_1 decreases until, for sufficiently large values of s_2, r_2 returns to zero.

V. CONCLUSIONS

In this article, we have summarized key findings from our work on two central aspects of a Brownian Motion with stochastic resetting in two states: the long-term behavior and the first-passage problem. We began by deriving the Fokker-Planck equations that characterize these processes using their Markovian properties. From these equations, we obtained the stationary state density function, along with an example illustrating the density's behavior across different parameter sets, highlighting how various parameters influence the density function.

For the first-passage problem, we derived a general expectation equation for the process and simplified the analysis by reducing the dynamics to a case where the two resetting points coincide, allowing for more direct comparison. We established that the expectation for the process is bounded by the expectations for single-diffusion cases, and we examined the monotonic behavior of $T(s_i)$ while keeping s_j fixed. We then analyzed how the expectation behaves relative to the values of r_1 and r_2, as well as the roles of the parameters s_1, s_2, and α. Notably, we found that while the minimum candidates for the expectation coincide when $\alpha = 1$, this is not the case for $\alpha \neq 1$.

Our main results reveal that the resetting rates exhibit two distinct states separated by a well-defined transition, depending on the relationships between α, s_1, and s_2. These states correspond to cases where r_2 tends to infinity and where r_2 remains finite.

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